SPECTRAL SHIFT FUNCTION AND RESONANCES NEAR THE LOW GROUND STATE FOR PAULI AND SCHRÖDINGER OPERATORS

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Abstract. We study the spectral shift function (SSF) $\xi(\lambda)$ and the resonances of the operator $H_V := \big(\sigma \cdot (-i\nabla - \mathbf{A})\big)^2 + V$ in $L^2(\mathbb{R}^3)$ near the origin. Here $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ are the 2×2 Pauli matrices and V is a hermitian potential decaying exponentially in the direction of the magnetic field $\mathbf{B} := \operatorname{curl} \mathbf{A}$. We give a representation of the derivative of the SSF as a sum of the imaginary part of a holomorphic function and a harmonic measure related to the resonances of H_V. This representation warrant the Breit-Wigner approximation moreover we deduce information about the singularities of the SSF at the origin and a local trace formula.

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Contents

1. Introduction and motivations

1.1. **Unperturbed operator.** Consider the three-dimensional Pauli operator acting in $L^2(\mathbb{R}^3) := L^2(\mathbb{R}^3, \mathbb{C}^2)$ and describing a quantum non-relativistic spin-$\frac{1}{2}$ particle subject to a magnetic field $\mathbf{B} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ pointing at the x_3 direction:

$$\mathbf{B}(\mathbf{x}) = \big(0, 0, b(\mathbf{x})\big), \quad \mathbf{x} := (x_\perp, x_3) \in \mathbb{R}^3, \quad x_\perp := (x_1, x_2) \in \mathbb{R}^2. \tag{1.1}$$

Then $x_\perp = (x_1, x_2) \in \mathbb{R}^2$ are the variables on the plane perpendicular to the magnetic field. Let $\mathbf{A} = (a_1, a_2, a_3) : \mathbb{R}^3 \to \mathbb{R}^3$ denote the magnetic potential generating the magnetic field, namely $\mathbf{B}(\mathbf{x}) := \operatorname{curl} \mathbf{A}(\mathbf{x})$. Since $\operatorname{div} \mathbf{B} = 0$ then b is independent of x_3. Hence there is no loss of generality in assuming that a_j, $j = 1, 2$ are independent of x_3 and $a_3 = 0$:

$$\mathbf{A}(\mathbf{x}) = \big(a_1(x_\perp), a_2(x_\perp), 0\big), \quad b(\mathbf{x}) = b(x_\perp) = \partial_1 a_2(x_\perp) - \partial_2 a_1(x_\perp). \tag{1.2}$$

Let σ_j, $j \in \{1, 2, 3\}$ be the 2×2 Pauli matrices given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

The free self-adjoint Pauli operator is initially defined on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ (then closed in $L^2(\mathbb{R}^3)$) by

$$H_0 := \big(\sigma \cdot (-i\nabla - \mathbf{A})\big)^2, \qquad \sigma := (\sigma_1, \sigma_2, \sigma_3). \tag{1.4}$$

A trivial computation shows that

$$H_0 = \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix}. \tag{1.5}$$

We will assume (abusing the terminology) that $b : \mathbb{R}^2 \to \mathbb{R}$ is an admissible magnetic field. This means that there exists a positive constant b_0 satisfying $b(x_\perp) = b_0 + \tilde{b}(x_\perp)$, $\tilde{b}$ being a function such that the Poisson equation

$$\Delta \tilde{\varphi} = \tilde{b} \tag{1.6}$$

admits a solution $\tilde{\varphi} \in C^2(\mathbb{R}^2)$ verifying $\sup_{x_\perp \in \mathbb{R}^2} |D^\alpha \tilde{\varphi}(x_\perp)| < \infty$, $\alpha \in \mathbb{Z}_+^2$, $|\alpha| \leq 2$, (we refer to [18, Section 2.1] for more details and examples on admissible magnetic fields). Introduce $\varphi_0(x_\perp) = b_0 |x_\perp|^2/4$ and $\varphi := \varphi_0 + \tilde{\varphi}$ so that we have $\Delta \varphi = b$. Define originally on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$ the operators

$$a = a(b) := -2i\mathrm{e}^{-\varphi} \frac{\partial}{\partial \bar{z}} \mathrm{e}^{\varphi} \quad \text{and} \quad a^* = a^*(b) := -2i\mathrm{e}^{\varphi} \frac{\partial}{\partial z} \mathrm{e}^{-\varphi} \tag{1.7}$$

with $z := x_1 + ix_2$, $\bar{z} := x_1 - ix_2$ and introduce the operators

$$H_1(b) = a^* a \quad \text{and} \qquad H_2(b) = a a^*. \tag{1.8}$$

The spectral properties of $H_j = H_j(b)$, $j = 1, 2$ are well know from [18, Proposition 1.1]:

$$\begin{cases} \sigma(H_1) \subseteq \{0\} \cup [\zeta, +\infty) \text{ whith } 0 \text{ an eigenvalue of infinite multiplicity,} \\ \sigma(H_2) \subseteq [\zeta, +\infty), \quad \dim \operatorname{Ker} H_2 = 0, \end{cases} \tag{1.9}$$

where

$$\zeta := 2b_0 e^{-2\operatorname{osc}\tilde{\varphi}}, \quad \operatorname{osc}\tilde{\varphi} := \sup_{x_\perp \in \mathbb{R}^2} \tilde{\varphi}(x_\perp) - \inf_{x_\perp \in \mathbb{R}^2} \tilde{\varphi}(x_\perp). \tag{1.10}$$

The orthogonal projection onto $\operatorname{Ker} H_1(b)$ will be denoted by $p = p(b)$. From [11, Theorem 2.3] we know that it admits a continuous integral kernel $\mathcal{P}_b(x_\perp, x'_\perp)$, $x_\perp$, $x'_\perp \in \mathbb{R}^2$. Furthermore by [18, Lemma 2.3]

$$\frac{b_0}{2\pi} e^{-2\operatorname{osc}\tilde{\varphi}} \leq \mathcal{P}_b(x_\perp, x_\perp) \leq \frac{b_0}{2\pi} e^{2\operatorname{osc}\tilde{\varphi}}, \qquad x_\perp \in \mathbb{R}^2. \tag{1.11}$$

Under the above considerations by taking $a_1 = -\partial_2\varphi$ and $a_2 = \partial_1\varphi$ the operator H_0 can be written in $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$ as

$$H_0 = \begin{pmatrix} H_1(b) \otimes 1 + 1 \otimes \left(-\frac{d^2}{dx_3^2}\right) & 0 \\ 0 & H_2(b) \otimes 1 + 1 \otimes \left(-\frac{d^2}{dx_3^2}\right) \end{pmatrix} =: \begin{pmatrix} \mathcal{H}_1(b) & 0 \\ 0 & \mathcal{H}_2(b) \end{pmatrix}. \tag{1.12}$$

The spectrum of $-\frac{d^2}{dx_3^2}$ originally defined on $C_0^\infty(\mathbb{R}, \mathbb{C})$ coincides with $[0, +\infty)$ and is absolutely continuous. Then (1.9) and (1.12) imply that

$$\sigma(H_0) = \sigma_{\mathbf{ac}}(H_0) = [0, +\infty), \tag{1.13}$$

$\big($see also [18, Corollary 2.2]$\big)$.

1.2. **Perturbed operator and the spectral shift function.** On the domain of H_0 we introduce the perturbed operator

$$H_V := H_0 + V, \tag{1.14}$$

where V is identified with the multiplication operator by the matrix-valued function

$$V(\mathbf{x}) := \begin{pmatrix} v_{11}(\mathbf{x}) & v_{12}(\mathbf{x}) \\ v_{21}(\mathbf{x}) & v_{22}(\mathbf{x}) \end{pmatrix} \in \mathfrak{B}_h(\mathbb{C}^2), \quad \mathbf{x} \in \mathbb{R}^3, \tag{1.15}$$

$\mathfrak{B}_h(\mathbb{C}^2)$ being the set of 2×2 hermitian matrices. Throughout this paper we require an exponential decay along the direction of the magnetic field for the electric potential V in the following sense:

$$\begin{cases} 0 \not\equiv V \in C^0(\mathbb{R}^3), \quad |v_{\ell k}(\mathbf{x})| \leq \text{Const.}\, \langle x_\perp \rangle^{-m_\perp} e^{-\gamma\langle x_3 \rangle}, \quad 1 \leq \ell, k \leq 2 \\ \text{with } m_\perp > 2, \gamma > 0 \text{ constant and } \langle y \rangle := \sqrt{1 + |y|^2} \text{ for } y \in \mathbb{R}^d. \end{cases} \tag{1.16}$$

Introduce some notations. Let $\mathscr{H}$ be a separable Hilbert space and $\mathcal{S}_\infty(\mathscr{H})$ be the set of compact linear operators on $\mathscr{H}$. Denote by $s_k(T)$ the k-th singular value of $T \in \mathcal{S}_\infty(\mathscr{H})$.

The Schatten-von Neumann class ideals $\mathcal{S}_q(\mathscr{H})$, $q \in [1, +\infty)$ are defined by

$$\mathcal{S}_q(\mathscr{H}) := \Big\{ T \in \mathcal{S}_\infty(\mathscr{H}) : \|T\|_{\mathcal{S}_q}^q := \sum_k s_k(T)^q < +\infty \Big\}. \tag{1.17}$$

For $\lceil q \rceil := \min \big\{ n \in \mathbb{N} : n \geq q \big\}$ and $T \in \mathcal{S}_q(\mathscr{H})$ the regularized determinant $\det_{\lceil q \rceil}(I - T)$ is defined by

$$\det{}_{\lceil q \rceil}(I - T) := \prod_{\mu \, \in \, \sigma(T)} \left[(1 - \mu) \exp \left(\sum_{k=1}^{\lceil q \rceil - 1} \frac{\mu^k}{k} \right) \right]. \tag{1.18}$$

The case $q = 1$ corresponds to the trace class operators while the case $q = 2$ coincides with the Hilbert-Schmidt operators.

Now let $\mathcal{H}_0$ and $\mathcal{H}$ be two self-adjoint operators in $\mathscr{H}$ such that

$$V := \mathcal{H} - \mathcal{H}_0 \in \mathcal{S}_1(\mathscr{H}). \tag{1.19}$$

There exists an important object in the theory of scattering associated to the pair of operators $(\mathcal{H}, \mathcal{H}_0)$ called the *spectral shift function* (SSF) $\xi(\lambda)$. The concept of SSF was first formally introduced by Lifshits [16]. The mathematical theory of the SSF was developed by Krein [14]. For trace class perturbations (1.19) the SSF is related to the determinant perturbation by the Krein's formula $\big($see for instance [14], [15]$\big)$

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \longrightarrow 0^+} \operatorname{Arg} \det \big(I + V(\mathcal{H}_0 - \lambda - i\varepsilon)^{-1} \big), \quad \text{a.e. } \lambda \in \mathbb{R}, \tag{1.20}$$

the branch of the argument being fixed by the condition

$$\operatorname{Arg} \det \big(I + V(\mathcal{H}_0 - z)^{-1} \big) \longrightarrow 0, \quad \operatorname{Im}(z) \longrightarrow +\infty.$$

Actually on the basis of the invariance principle $\big($see for instance [3]$\big)$ the SSF is well defined once there exists $\ell > 0$ such that

$$(\mathcal{H} - i)^{-\ell} - (\mathcal{H}_0 - i)^{-\ell} \in \mathcal{S}_1(\mathscr{H}). \tag{1.21}$$

It's the function whose derivative is given by the following distribution:

$$\xi' : f \longmapsto -\operatorname{Tr}\big(f(\mathcal{H}) - f(\mathcal{H}_0) \big), \quad f \in C_0^\infty(\mathbb{R}). \tag{1.22}$$

Following the Birman-Krein theory $\big($see [2]$\big)$ the SSF coincides with the scattering phase $s(\lambda) = -\frac{1}{2\pi} \operatorname{Arg} \det S(\lambda)$ where $S(\lambda)$ is the scattering matrix. More precisely by the Birman-Krein formula $\big($see [2]$\big)$ the SSF is related to $S(\lambda)$ by $\det S(\lambda) = e^{-2i\pi S(\lambda)}$ for almost every $\lambda \in \sigma_{ac}(H_0)$. The above interpretation of the SSF as the scattering phase stimulates its investigation in quantum-mechanical problems. We refer to the review [3] and the book [24] for a large detailed bibliography about the SSF.

In our case assumption (1.16) on V implies that there exists $\mathscr{V} \in \mathscr{L}(\mathscr{H})$ such that

$$|V|^{\frac{1}{2}}(\mathbf{x}) = \mathscr{V} \left(\langle x_\perp \rangle^{-\frac{m_\perp}{2}} \otimes e^{-\frac{\gamma}{2}\langle t \rangle} \right), \quad \mathbf{x} = (x_\perp, t) \in \mathbb{R}^3, \quad m_\perp > 2. \tag{1.23}$$

The standard criterion [20, Theorem 4.1] implies that

$$\langle x_\perp \rangle^{-\frac{m_\perp}{2}} \otimes e^{-\frac{\gamma}{2}\langle t \rangle}(-\Delta + 1)^{-1} \in \mathcal{S}_2\big(L^2(\mathbb{R}^3, \mathbb{C})\big). \tag{1.24}$$

Then this together with the diamagnetic inequality $\big($see [1, Theorem 2.3]-[20, Theorem 2.13]$\big)$ and the boundedness of the magnetic field b imply that

$$|V|^{\frac{1}{2}}(H_0 - i)^{-1} \in \mathcal{S}_2\big(L^2(\mathbb{R}^3)\big). \tag{1.25}$$

Therefore exploiting the resolvent identity we obtain

$$(H_V - i)^{-1} - (H_0 - i)^{-1} \in \mathcal{S}_1\big(L^2(\mathbb{R}^3)\big). \tag{1.26}$$

Namely (1.21) holds with $\ell = 1$ with respect to the operators H_V, H_0 and the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3)$. So the distribution

$$\xi' : f \longmapsto -\mathrm{Tr}\big(f(H_V) - f(H_0)\big), \quad f \in C_0^\infty(\mathbb{R}) \tag{1.27}$$

is well defined. For our purpose it is more convenient to introduce the regularized spectral shift function $\big($see for instance [13] or [4]$\big)$

$$\xi_2(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \longrightarrow 0^+} \mathrm{Arg}\, \mathrm{det}_2\big(I + V(H_0 - \lambda - i\varepsilon)^{-1}\big) \tag{1.28}$$

whose derivative is given by the distribution

$$\xi_2' : f \longmapsto -\mathrm{Tr}\left(f(H_V) - f(H_0) - \frac{d}{d\varepsilon} f(H_0 + \varepsilon V)_{|\varepsilon=0}\right), \quad f \in C_0^\infty(\mathbb{R}). \tag{1.29}$$

From the relation between ξ' and ξ_2' given by Lemma 5.1, we will deduce the properties of the SSF. In the present paper the main result concerns a representation of the derivative of the SSF near the low ground state of the operator H_0 corresponding to the origin as a sum of a harmonic measure (related to the resonances of the operator H_V near zero) and the imaginary part of a holomorphic function. Such representation justifies the Breit-Wigner approximation (see Theorem 2.1) and implies a trace formula (see Theorem 2.2) as in [17], [7], [9], [5]. We derive also from our main result an asymptotic expansion of the SSF near the origin (see Theorem 2.3). Similar results are obtained in [5] for the SSF near the Landau levels as well in [12]. On the other hand the singularities of the SSF associated to the pair (H_V, H_0) is also studied in [18] with polynomial decay on the electric potential V. In Remark 2.2, we compare our results to those of [18]. The case of the Dirac Hamiltonian with admissible magnetic fields is considered in [23] where the singularities of the SSF near $\pm m$ are investigated. Results obtained there are closely related to those from [18].

The paper is organized as follows. In Section 2 we formulate our main results. Sections 3-4 are devoted to the study of the resonances of H_V near the origin. In the first one we define the resonances and in the second one we establish upper bounds on their number near the origin. Sections 5-7 are respectively devoted to the proofs of the main results. Section 8 is a brief appendix on finite meromorphic operator-valued functions.

2. Statement of the main results

First introduce some notations and terminology.

Denote by $|V|$ the multiplication operator by the matrix-valued function

$$\sqrt{V^*V}(\mathbf{x}) = \sqrt{V^2}(\mathbf{x}) =: \big\{|V|_{\ell k}(\mathbf{x})\big\}, \quad 1 \leq \ell, k \leq 2 \tag{2.1}$$

and by $J := sign(V)$ the matrix sign of V which satisfies $V = J|V|$. We will say that V is of definite sign if the multiplication operator $V(\mathbf{x})$ by the matrix-valued function $V(\mathbf{x})$ satisfies

$$\pm\, V(\mathbf{x}) \geq 0 \tag{2.2}$$

for any $\mathbf{x} \in \mathbb{R}^3$. It is easy to check that in this case we have respectively $V = J|V| = \pm|V|$. Then without loss of generality we will say that V is of definite sign $J = \pm$.

Let $\mathbf{W}$ be the multiplication operator by the function $\mathbf{W} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$\mathbf{W}(x_\perp) := \int_{\mathbb{R}} |V|_{11}(x_\perp, x_3) dx_3. \tag{2.3}$$

Hypothesis (1.16) on V implies that

$$0 \leq \mathbf{W}(x_\perp) \leq \mathrm{Const.}' \langle x_\perp \rangle^{-m_\perp}, \quad m_\perp > 2, \quad x_\perp \in \mathbb{R}^2, \tag{2.4}$$

where $\mathrm{Const.}' = \mathrm{Const.} \int_{\mathbb{R}} e^{-\gamma \langle x_3 \rangle} dx_3$. Then by [18, Lemma 2.3] the positive self-adjoint Toeplitz operator $p\mathbf{W}p$ is of trace class, $p = p(b)$ being the orthogonal projection onto $\mathrm{Ker}\, H_1(b)$ defined by (1.8).

Introduce $e_\pm$ the multiplication operators by the functions $e^{\pm \frac{\gamma}{2} \langle \cdot \rangle}$ respectively and let $c : L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ be the operator given by

$$c(u) := \langle u, e^{-\frac{\gamma}{2} \langle \cdot \rangle} \rangle \tag{2.5}$$

while $c^* : \mathbb{C} \longrightarrow L^2(\mathbb{R})$ satisfies $c^*(\lambda) = \lambda e^{-\frac{\gamma}{2} \langle \cdot \rangle}$. Define the operator $K : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^2)$ by

$$K := \frac{1}{\sqrt{2}} (p \otimes c) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e_+ |V|^{\frac{1}{2}}. \tag{2.6}$$

To be more explicit we have

$$(K\psi)(\mathbf{x}) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \mathcal{P}_b(x_\perp, x'_\perp) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} (x'_\perp, x'_3) \psi(x'_\perp, x'_3) dx'_\perp dx'_3, \tag{2.7}$$

where $\mathcal{P}_b(\cdot, \cdot)$ is the integral kernel of the orthogonal projection p. Obviously the adjoint operator $K^* : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^3)$ verifies

$$(K^*\varphi)(x_\perp, x_3) = \frac{1}{\sqrt{2}} |V|^{\frac{1}{2}} (x_\perp, x_3) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (p\varphi)(x_\perp). \tag{2.8}$$

Then

$$KK^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{p\mathbf{W}p}{2} \tag{2.9}$$

so that it is a self-adjoint positif compact operator.

Now let us introduce technical important conditions. Define the constant

$$N_{\gamma,\zeta} := \min\left(\frac{\gamma}{2}, \sqrt{\zeta}\right), \tag{2.10}$$

where γ and ζ are respectively defined by (1.16) and (1.10). Let $\mathscr{W}_\pm \Subset \Omega_\pm$ be open relatively compact subsets of $\pm]0, N^2_{\gamma,\zeta}[e^{\pm i]-2\theta_0,2\varepsilon_0[}$ such that $0 < \min(\theta_0, \varepsilon_0)$ and $\max(\theta_0, \varepsilon_0) < \frac{\pi}{2}$. Let $r > 0$ be a small parameter and assume that $\mathscr{W}_\pm$ and $\Omega_\pm$ are simply connected sets independent of r. We also assume that the intersections between $\pm]0, N^2_{\gamma,\zeta}[$ and $\mathscr{W}_\pm$, $\Omega_\pm$ are intervals. Hence we set $I_\pm := \mathscr{W}_\pm \cap \pm]0, N^2_{\gamma,\zeta}[$.

In the case where the potential V is of definite sign $J = sign(V)$ the representation of the SSF near zero can be specified. This required firstly that for $k \in \mathbb{C}$ small enough the operator $I + \frac{iJ}{k}K^*K$ be invertible. That is for $\operatorname{Arg} k \neq -J\frac{\pi}{2}$. Secondly that the condition

$$-J\frac{\pi}{2} \notin \left(\frac{\pi}{2}\right)_\mp \pm [-\theta_0, \varepsilon_0] \tag{2.11}$$

be satisfied with respect to the subscript "$\pm$" in $\mathscr{W}_\pm \Subset \Omega_\pm$, $I_\pm := \mathscr{W}_\pm \cap \pm]0, N^2_{\gamma,\zeta}[$, where $\left(\frac{\pi}{2}\right)_- = 0$ and $\left(\frac{\pi}{2}\right)_+ = \frac{\pi}{2}$.

Remark 2.1. –

(i) *Under our considerations on θ_0 and ε_0 above condition* (2.11) *is satisfied in the case "$+$" for $J = \pm$. Namely*

$$\mp\frac{\pi}{2} \notin [-\theta_0, \varepsilon_0], \quad J = \pm. \tag{2.12}$$

(ii) *In the case "$-$" condition* (2.11) *is satisfied for $J = +$. Namely*

$$-\frac{\pi}{2} \notin \left[\frac{\pi}{2} - \varepsilon_0, \frac{\pi}{2} + \theta_0\right]. \tag{2.13}$$

From now on the set of the resonances near zero of H_V (see Definition 3.1) will be denoted by $\operatorname{Res}(H_V)$. Our first main result goes as follows:

Theorem 2.1. (Breit-Wigner approximation)

Assume that assumption (1.16) *holds. Let $\mathscr{W}_\pm \Subset \Omega_\pm$ be open relatively compact subsets of $\pm]0, N^2_{\gamma,\zeta}[e^{\pm i]-2\theta_0,2\varepsilon_0[}$ as above. Choose moreover $0 < s_1 < \sqrt{\operatorname{dist}(\Omega_\pm, 0)}$. Then there exists $r_0 > 0$ and holomorphic functions $g_\pm$ in $\Omega_\pm$ satisfying for any $\mu \in rI_\pm$ and $r < r_0$*

$$\xi'(\mu) = \frac{1}{r\pi}\operatorname{Im} g'_\pm\left(\frac{\mu}{r}, r\right) + \sum_{\substack{w \in \operatorname{Res}(H_V) \cap r\Omega_\pm \\ \operatorname{Im}(w) \neq 0}} \frac{\operatorname{Im}(w)}{\pi|\mu - w|^2} - \sum_{w \in \operatorname{Res}(H_V) \cap rI_\pm} \delta(\mu - w), \tag{2.14}$$

where the functions $g_\pm(z, r)$ satisfy the bound

$$\begin{aligned} g_\pm(z, r) &= \mathcal{O}\left[\operatorname{Tr} \mathbf{1}_{(s_1\sqrt{r},\infty)}(p\mathbf{W}p)|\ln r| + \tilde{n}_1\left(\frac{1}{2}s_1\sqrt{r}\right) + \tilde{n}_2\left(\frac{1}{2}s_1\sqrt{r}\right)\right] \\ &= \mathcal{O}\left(|\ln r| r^{-1/m_\perp}\right), \end{aligned} \tag{2.15}$$

uniformly with respect to $0 < r < r_0$ *and* $z \in \Omega_\pm$*, with* $\tilde{n}_q(\cdot)$*,* $q = 1,\, 2$ *defined by* (4.24)*.*

Furthermore for potentials of definite sign $J = sign(V)$ *we have for* $\lambda \in rI_\pm$

$$\frac{1}{r}\mathrm{Im}\, g'_\pm\left(\frac{\lambda}{r}, r\right) = \frac{1}{r}\mathrm{Im}\, \tilde{g}'_\pm\left(\frac{\lambda}{r}, r\right) + \mathrm{Im}\, \tilde{g}'_{1,\pm}(\lambda) + \mathbf{1}_{(0,N^2_{\gamma,\varsigma})}(\lambda) J\phi'(\lambda), \tag{2.16}$$

where the function ϕ *is defined by*

$$\phi(\lambda) := \mathrm{Tr}\left(\arctan\frac{K^*K}{\sqrt{\lambda}}\right) = \mathrm{Tr}\left(\arctan\frac{p\mathbf{W}p}{2\sqrt{\lambda}}\right), \tag{2.17}$$

the functions $z \mapsto \tilde{g}_\pm(z, r)$ *being holomorphic in* $\Omega_\pm$ *and satisfying*

$$\tilde{g}_\pm(z, r) = \mathcal{O}\big(|\ln r|\big), \tag{2.18}$$

uniformly with respect to $0 < r < r_0$ *and* $z \in \Omega_\pm$*. The functions* $z \mapsto \tilde{g}_{1,\pm}(z)$ *are holomorphic in* $\pm]0, N^2_{\gamma,\varsigma}[e^{\pm i]-2\theta_0,2\varepsilon_0[}$ *and there exists a positive constant* C_{θ_0} *depending on* θ_0 *such that*

$$|\tilde{g}_{1,\pm}(z)| \leq C_{\theta_0}\sigma_2\left(\sqrt{|z|}\right)^{\frac{1}{2}} \tag{2.19}$$

for $z \in \pm]0, N^2_{\gamma,\varsigma}[e^{\pm i]-2\theta_0,2\varepsilon_0[}$*, where the quantity* $\sigma_2(\cdot)$ *is defined by* (4.22)*.*

As first consequence of the above theorem we have the following result describing the asymptotic behaviour of the SSF on the right of the low ground state.

Theorem 2.2. (Singularity at the low ground state)

Assume that V *satisfies assumption* (1.16) *with definite sign* $J = sign(V)$*. Then*

$$\xi(\lambda) = \frac{J}{\pi}\phi(\lambda) + \mathcal{O}\left(\phi(\lambda)^{\frac{1}{2}}\right) + \mathcal{O}\big(|\ln\lambda|^2\big) \tag{2.20}$$

as $\lambda \searrow 0$*, the function* $\phi(\lambda)$ *being defined by* (2.17)*.*

Remark 2.2. –

(i) *Since for* $\lambda > 0$

$$\xi(-\lambda) = -\#\big\{\textit{discrete eigenvalues of } H_V \textit{ lying in } (-\infty, -\lambda)\big\} \tag{2.21}$$

then for $V \geq 0$ *we have* $\xi(-\lambda) = 0$*.*

(ii) *In* [18] *the singularities of the SSF near the origin are studied. If* $\mathbf{W}$ *satisfies assumptions* ***(A1)****,* ***(A2)*** *or* ***(A3)*** *implying respectively* (4.16)*,* (4.17) *or* (4.18) *then it is proved in* [18] *that*

$$\xi(\lambda) = \frac{J}{\pi}\phi(\lambda)\big(1 + o(1)\big), \quad \lambda \searrow 0. \tag{2.22}$$

Thus (2.20) *provides a remainder estimate of* (2.22) *when* $\mathbf{W}$ *satisfies assumption* ***(A1)****. However for* $V \leq 0$ *it is proved in* [18] *that*

$$\xi(-\lambda) = -\mathrm{Tr}\,\mathbf{1}_{(2\sqrt{\lambda},\infty)}\big(p\mathbf{W}p\big)\big(1 + o(1)\big), \quad \lambda \searrow 0. \tag{2.23}$$

As second consequence of Theorem 2.1 we have the following

Theorem 2.3. (Local trace formula)

Let the domains $\mathscr{W}_\pm \Subset \Omega_\pm$ *be as in Theorem 2.1. Assume that* $f_\pm$ *is holomorphic in a neighbourhood of* $\Omega_\pm$ *and let* $\psi_\pm \in C_0^\infty\big(\Omega_\pm \cap \mathbb{R}\big)$ *satisfy* $\psi_\pm(\lambda) = 1$ *near* $\Omega_\pm \cap \mathbb{R}$. *Then under the assumptions of Theorem 2.1*

$$\mathrm{Tr}\left[(\psi_\pm f_\pm)\left(\frac{H_V}{r}\right) - (\psi_\pm f_\pm)\left(\frac{H_0}{r}\right)\right] = \sum_{w \in \mathrm{Res}(H_V)\cap r\mathscr{W}_\pm} f_\pm\left(\frac{w}{r}\right) + E_{f_\pm,\psi_\pm}(r) \tag{2.24}$$

with

$$|E_{f_\pm,\psi_\pm}(r)| \leq M(\psi_\pm)\sup\big\{|f_\pm(z)| : z \in \Omega_\pm \setminus \mathscr{W}_\pm : \mathrm{Im}(z) \leq 0\big\} \times N(r), \tag{2.25}$$

where

$$\begin{aligned} N(r) &= \mathrm{Tr}\,\mathbf{1}_{(s_1\sqrt{r},\infty)}\big(p\mathbf{W}p\big)|\ln r| + \tilde{n}_1\left(\frac{1}{2}s_1\sqrt{r}\right) + \tilde{n}_2\left(\frac{1}{2}s_1\sqrt{r}\right) \\ &= \mathcal{O}\left(|\ln r| r^{-1/m_\perp}\right). \end{aligned} \tag{2.26}$$

Remark 2.3. (Schrödinger operator)

Our results remain true if instead the operator H_V *defined by* (1.14) *we consider in* $L^2(\mathbb{R}^3,\mathbb{C})$ *the perturbed Schrödinger operator*

$$(-i\nabla - \boldsymbol{A})^2 - b + V \tag{2.27}$$

on Dom$\big((-i\nabla - \boldsymbol{A})^2 - b\big)$ *with* $V(\mathbf{x}) = \mathcal{O}\left(\langle x_\perp\rangle^{-m_\perp} e^{-\gamma\langle x_3\rangle}\right)$ *for any* $\mathbf{x} \in \mathbb{R}^3$, $m_\perp > 2$, $\gamma > 0$ *as in* (1.16). *Here* $\mathbf{W}$ *is just given by* $\mathbf{W}(x_\perp) = \displaystyle\int_{\mathbb{R}} |V(x_\perp, x_3)|dx_3$ *for any* $x_\perp \in \mathbb{R}^2$ *and in identities* (2.6)-(2.9) *the matrix* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *is removed.*

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3. Definition of the resonances

The potential V is assumed to satisfy (1.16). We recall also that $p = p(b)$ is the orthogonal projection onto $\mathrm{Ker}\, H_1$ with $H_1 = H_1(b)$ defined by (1.8).

Set $P := p \otimes 1$, $Q := I - P$. Introduce the orthogonal projections in $L^2(\mathbb{R}^3)$

$$\mathrm{P} := \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathrm{Q} := \mathrm{I} - \mathrm{P} = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}. \tag{3.1}$$

For $z \in \mathbb{C} \setminus [0,+\infty)$ (1.14) and (1.9) imply that

$$(H_0 - z)^{-1}\mathrm{P} = \begin{pmatrix} p \otimes \mathscr{R}(z) & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.2}$$

where $\mathscr{R}(z) := \left(-\frac{d^2}{dt^2} - z\right)^{-1}$ acts in $L^2(\mathbb{R})$. Thus

$$(H_0 - z)^{-1} = \big(p \otimes \mathscr{R}(z)\big) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (H_0 - z)^{-1}\mathrm{Q}. \tag{3.3}$$

The one-dimensional resolvent $\mathscr{R}(z)$ introduced above admits the integral kernel

$$\mathscr{N}_z(t,t') = \frac{i\mathrm{e}^{i\sqrt{z}|t-t'|}}{2\sqrt{z}} \tag{3.4}$$

if the branch $\mathrm{Im}(\sqrt{z})$ is chosen such that $\mathrm{Im}(\sqrt{z}) > 0$. In the sequel we set

$$\mathbb{C}^+ := \big\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\big\} \quad \text{and} \quad \mathbb{C}^+_{1/2} := \big\{k \in \mathbb{C} : k^2 \in \mathbb{C}^+\big\}. \tag{3.5}$$

With respect to the variable k we define the pointed disk

$$D(0,\epsilon)^* := \big\{k \in \mathbb{C} : 0 < |k| < \epsilon\big\} \tag{3.6}$$

with

$$\epsilon < N_{\gamma,\zeta} \tag{3.7}$$

the constant defined by (2.10).

In order to define the resonances near zero first we extend holomorophically $(H_0 - k^2)^{-1}\mathrm{P}$ near $k = 0$.

Proposition 3.1. *Let $\gamma > 0$ be constant and set $z(k) := k^2$.*

(i) *The operator valued-function*

$$k \longmapsto \Big(\big(H_0 - z(k)\big)^{-1}\mathrm{P} : e^{-\frac{\gamma}{2}\langle t\rangle} L^2(\mathbb{R}^3) \longrightarrow e^{\frac{\gamma}{2}\langle t\rangle} L^2(\mathbb{R}^3)\Big)$$

admits a holomorphic extension from $\mathbb{C}^+_{1/2} \cap D(0,\epsilon)^*$ to $D(0,\epsilon)^*$.

(ii) *For $v_\perp(x_\perp) := \langle x_\perp\rangle^{-\alpha}$ with $\alpha > 1$ the operator valued-function*

$$T_{v_\perp} : k \longmapsto v_\perp(x_\perp) e^{-\frac{\gamma}{2}\langle t\rangle}\big(H_0 - z(k)\big)^{-1}\mathrm{P}e^{-\frac{\gamma}{2}\langle t\rangle}$$

has a holomorphic extension to $D(0,\epsilon)^*$ with values in the Hilbert-Schmidt class $\mathcal{S}_2\left(L^2(\mathbb{R}^3)\right)$.

Proof. **(i)** Introduce

$$L(k) = \big[p \otimes \mathcal{R}(k^2)\big] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.8}$$

acting from $e^{-\frac{\gamma}{2}\langle t\rangle} L^2(\mathbb{R}^3)$ to $e^{\frac{\gamma}{2}\langle t\rangle} L^2(\mathbb{R}^3)$. The operator $\mathscr{N}(k) := e^{-\frac{\gamma}{2}\langle t\rangle}\mathscr{R}(k^2)e^{-\frac{\gamma}{2}\langle t\rangle}$ admits the integral kernel

$$e^{-\frac{\gamma}{2}\langle t\rangle}\frac{i\mathrm{e}^{ik|t-t'|}}{2k}\mathrm{e}^{-\frac{\gamma}{2}\langle t'\rangle}. \tag{3.9}$$

It is easy to check that the integral kernel (3.9) belongs to $L^2(\mathbb{R})$ once $\mathrm{Im}(k) > -\frac{\gamma}{2}$, $k \in \mathbb{C}^*$. Then for $\epsilon < \frac{\gamma}{2}$ we can extend holomorphically $k \longmapsto L(k) \in \mathscr{L}\left(e^{-\frac{\gamma}{2}\langle t\rangle} L^2(\mathbb{R}^3), e^{\frac{\gamma}{2}\langle t\rangle} L^2(\mathbb{R}^3)\right)$

from $\mathbb{C}^+_{1/2} \cap D(0,\epsilon)^*$ to $D(0,\epsilon)^*$. This together with (3.2) imply that $k \longmapsto \big(H_0 - z(k)\big)^{-1}\mathrm{P} \in \mathscr{L}\left(e^{-\frac{\gamma}{2}\langle t\rangle}L^2(\mathbb{R}^3), e^{\frac{\gamma}{2}\langle t\rangle}L^2(\mathbb{R}^3)\right)$ admits a holomorphic extension to $D(0,\epsilon)^*$.

(ii) Thanks to (3.2)

$$T_{v_\perp}(k) = [v_\perp p \otimes \mathscr{N}(k)] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.10}$$

The operator $\mathscr{N}(k) \in \mathcal{S}_2\big(L^2(\mathbb{R})\big)$ following the proof of assertion **(i)** for $\mathrm{Im}(k) > -\frac{\gamma}{2}$, $k \in \mathbb{C}^*$. Since $v_\perp^2 \in L^1(\mathbb{R}^2)$ then by [18, Lemma 2.3] $pv_\perp^2 p$ is a trace class operator in $L^2(\mathbb{R}^2)$. That is $v_\perp p v_\perp \in \mathcal{S}_1\big(L^2(\mathbb{R}^2)\big)$. This together with (1.11) imply that $v_\perp p \in \mathcal{S}_2\big(L^2(\mathbb{R}^2)\big)$ with

$$\|v_\perp p\|^2_{\mathcal{S}_2} = \mathrm{Tr}\,(v_\perp p v_\perp) = \int_{\mathbb{R}^2} v_\perp^2(x_\perp)\mathcal{P}_b(x_\perp, x_\perp)dx_\perp \leq \frac{b_0}{2\pi}e^{2\mathrm{osc}\,\tilde{\varphi}} \int_{\mathbb{R}^2} v_\perp^2(x_\perp)dx_\perp. \tag{3.11}$$

Thus $k \mapsto T_{v_\perp}(k)$ has a holomorphic extension as above from $\mathbb{C}^+_{1/2} \cap D(0,\epsilon)^*$ to $D(0,\epsilon)^*$ with values in $\mathcal{S}_2\left(L^2(\mathbb{R}^3)\right)$. The proof is complete. □

Now let us extend holomorphically the operator $(H_0 - z)^{-1}\mathrm{Q}$ from the upper half-plane to the lower half-plane except a semi-axis.

Proposition 3.2. *Let γ be as in Proposition 3.1 and ζ be defined by* (1.10).

(i) *The operator valued-function*

$$z \longmapsto \Big((H_0 - z)^{-1}\mathrm{Q} : e^{-\frac{\gamma}{2}\langle t\rangle}L^2(\mathbb{R}^3) \longrightarrow e^{\frac{\gamma}{2}\langle t\rangle}L^2(\mathbb{R}^3)\Big)$$

admits a holomorphic extension from $\mathbb{C}^+$ to $\mathbb{C} \setminus [\zeta, \infty)$.

(ii) *For $v_\perp(x_\perp) := \langle x_\perp\rangle^{-\alpha}$ with $\alpha > 1$ the operator valued-function*

$$L_{v_\perp} : z \longmapsto v_\perp(x_\perp)e^{-\frac{\gamma}{2}\langle t\rangle}(H_0 - z)^{-1}\mathrm{Q}e^{-\frac{\gamma}{2}\langle t\rangle}$$

has a holomorphic extension to $\mathbb{C}\setminus[\zeta,\infty)$ with values in the Hilbert-Schmidt class $\mathcal{S}_2\left(L^2(\mathbb{R}^3)\right)$.

Proof. **(i)** Consider $z \in \mathbb{C}^+$. Thanks to (1.12) and (3.1) we have

$$(H_0 - z)^{-1}\mathrm{Q} = \begin{pmatrix} \big(\mathcal{H}_1(b)-z\big)^{-1}Q & 0 \\ 0 & \big(\mathcal{H}_2(b)-z\big)^{-1} \end{pmatrix} = \big(\mathcal{H}_1(b) - z\big)^{-1}Q \oplus \big(\mathcal{H}_2(b) - z\big)^{-1}. \tag{3.12}$$

Since $\mathbb{C} \setminus [\zeta,\infty)$ is contained in the resolvent set of $\mathcal{H}_1(b)$ acting on $QDom\big(\mathcal{H}_1(b)\big)$ and $\mathcal{H}_2(b)$ acting on $Dom\big(\mathcal{H}_2(b)\big)$ then $\mathbb{C} \setminus [\zeta,\infty) \ni z \longmapsto \big(\mathcal{H}_1(b) - z\big)^{-1}Q \oplus \big(\mathcal{H}_2(b) - z\big)^{-1}$ is well defined and holomorphic. So $\mathbb{C}^+ \ni z \mapsto e^{-\frac{\gamma}{2}\langle t\rangle}(H_0 - z)^{-1}\mathrm{Q}e^{-\frac{\gamma}{2}\langle t\rangle}$ admits a holomorphic extension to $\mathbb{C} \setminus [\zeta,\infty)$.

(ii) According to (3.12)

$$L_{v_\perp}(z) = v_\perp e^{-\frac{\gamma}{2}\langle t\rangle}\left(\big(\mathcal{H}_1(b) - z\big)^{-1}Q \oplus \big(\mathcal{H}_2(b) - z\big)^{-1}\right)e^{-\frac{\gamma}{2}\langle t\rangle}. \tag{3.13}$$

We have

(3.14)
$$\left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \big(\mathcal{H}_1(b)-z\big)^{-1} Q \right\|_{\mathcal{S}_2}^2 \leq \left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \big(\mathcal{H}_1(b)+1\big)^{-1} \right\|_{\mathcal{S}_2}^2 \left\| \big(\mathcal{H}_1(b)+1\big)\big(\mathcal{H}_1(b)-z\big)^{-1} Q \right\|^2.$$

By the Spectral mapping theorem

$$\left\| \big(\mathcal{H}_1(b)+1\big)\big(\mathcal{H}_1(b)-z\big)^{-1} Q \right\|^q \leq \sup_{s\in[\zeta,+\infty)}^q \left| \frac{s+1}{s-z} \right|. \tag{3.15}$$

With the help of the resolvent identity, the boundedness of the magnetic field b and the diamagnetic inequality $\big($see [1, Theorem 2.3]-[20, Theorem 2.13]$\big)$ we obtain

(3.16)
$$\begin{aligned} \left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \big(\mathcal{H}_1(b)+1\big)^{-1} \right\|_{\mathcal{S}_2}^2 &\leq \left\| I + \big(\mathcal{H}_1(b)+1\big)^{-1} b \right\|^2 \left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \big((-i\nabla - \mathbf{A})^2+1\big)^{-1} \right\|_{\mathcal{S}_2}^2 \\ &\leq C \left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} (-\Delta+1)^{-1} \right\|_{\mathcal{S}_2}^2. \end{aligned}$$

By the standard criterion [20, Theorem 4.1]

$$\left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} (-\Delta+1|)^{-1} \right\|_{\mathcal{S}_2}^2 \leq C \| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \|_{L^2}^2 \left\| \Big(|\cdot|^2+1 \Big)^{-1} \right\|_{L^2}^2. \tag{3.17}$$

Putting together (3.14), (3.15), (3.16) and (3.17) we get

$$\left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \big(\mathcal{H}_1(b)-z\big)^{-1} Q \right\|_{\mathcal{S}_2}^2 \leq C \| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \|_{L^2}^2 \sup_{s\in[\zeta,+\infty)}^2 \left| \frac{s+1}{s-z} \right|. \tag{3.18}$$

By similar arguments we can prove that

$$\left\| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \big(\mathcal{H}_2(b)-z\big)^{-1} \right\|_{\mathcal{S}_2}^2 \leq C \| v_\perp e^{-\frac{\gamma}{2}\langle t\rangle} \|_{L^2}^2 \sup_{s\in[\zeta,+\infty)}^2 \left| \frac{s+1}{s-z} \right|. \tag{3.19}$$

Since the multiplication operator by the function $e^{-\frac{\gamma}{2}\langle t\rangle}$ is bounded then (3.13), (3.18) and (3.19) imply that $L_{v_\perp}(z)$ belongs to $\mathcal{S}_2\left(L^2(\mathbb{R}^3)\right)$ and has a holomorphic extension from $\mathbb{C}^+$ to $\mathbb{C}\setminus[\zeta,\infty)$. This completes the proof. □

For V satisfying assumption (1.16), (1.23) holds. Then this together with (3.3), Propositions 3.1-3.2 yield to the following

Lemma 3.1. *Let $D(0,\epsilon)^*$ be the pointed disk defined by* (3.6). *Assume that V satisfies* (1.16) *and set $z(k) := k^2$. Then the operator valued-function*

$$\mathbb{C}_{1/2}^+ \cap D(0,\epsilon)^* \ni k \longmapsto \mathcal{T}_V\big(z(k)\big) := J|V|^{1/2}\big(H_0 - z(k)\big)^{-1}|V|^{1/2},$$

where $J := sign(V)$ has a holomorphic extension to $D(0,\epsilon)^*$ with values in $\mathcal{S}_2\left(L^2(\mathbb{R}^3)\right)$. We will denote again this extension by $\mathcal{T}_V\big(z(k)\big)$. Furthermore the operator $\partial_z \mathcal{T}_V\big(z(k)\big) \in \mathcal{S}_1\left(L^2(\mathbb{R}^3)\right)$ is holomorphic on $D(0,\epsilon)^*$.

Now using the identity

$$(H_V - z)^{-1}\big(1 + V(H_0 - z)^{-1}\big) = (H_0 - z)^{-1}$$

derived from the resolvent equation we obtain

$$\begin{aligned} e^{-\frac{\gamma}{2}\langle t\rangle}\big(H_V - z\big)^{-1} e^{-\frac{\gamma}{2}\langle t\rangle} &= e^{-\frac{\gamma}{2}\langle t\rangle}(H_0 - z)^{-1} e^{-\frac{\gamma}{2}\langle t\rangle} \\ &\quad \times \Big(1 + e^{\frac{\gamma}{2}\langle t\rangle} V (H_0 - z)^{-1} e^{-\frac{\gamma}{2}\langle t\rangle}\Big)^{-1}. \end{aligned}$$

As in (1.23) assumption (1.16) on V implies the existence of $\mathscr{M} \in \mathscr{L}\big(L^2(\mathbb{R}^3)\big)$ such that

$$|V|(x_\perp, t) = \mathscr{M}\left(\langle x_\perp\rangle^{-m_\perp} \otimes e^{-\gamma\langle t\rangle}\right), \quad (x_\perp, t) \in \mathbb{R}^3, \quad m_\perp > 2. \tag{3.20}$$

Then similarly to Lemma 3.1 it can be proved that $k \longmapsto e^{\frac{\gamma}{2}\langle t\rangle} V (H_0 - z)^{-1} e^{-\frac{\gamma}{2}\langle t\rangle}$ is holomorphic with values in $\mathcal{S}_\infty\left(L^2(\mathbb{R}^3)\right)$. Thus by the analytic Fredholm theorem the operator valued-function

$$k \longmapsto \Big(1 + e^{\frac{\gamma}{2}\langle t\rangle} V (H_0 - z)^{-1} e^{-\frac{\gamma}{2}\langle t\rangle}\Big)^{-1}$$

admits a meromorphic extension from $\mathbb{C}_{1/2}^+ \cap D(0,\epsilon)^*$ to $D(0,\epsilon)^*$. Hence we have the following

Proposition 3.3. *Under the assumptions and the notations of Lemma 3.1 the operator valued-function*

$$k \longmapsto \Big(\big(H_V - z(k)\big)^{-1} : \mathrm{e}^{-\frac{\gamma}{2}\langle x_3\rangle} L^2(\mathbb{R}^3) \longrightarrow \mathrm{e}^{\frac{\gamma}{2}\langle x_3\rangle} L^2(\mathbb{R}^3)\Big)$$

admits a meromorphic extension from $\mathbb{C}_{1/2}^+ \cap D(0,\epsilon)^*$ to $D(0,\epsilon)^*$. This extension will be denoted by $R\big(z(k)\big)$.

We can now define the resonances of H_V near zero. In the following definition the index of a finite-meromorphic operator valued-function appearing in (3.21) is recalled in the Appendix.

Definition 3.1. *We define the resonances of H near zero as the poles of the meromorphic extension $R(z)$ of $(H_V - z)^{-1}$ in $\mathscr{L}\left(\mathrm{e}^{-\frac{\gamma}{2}\langle x_3\rangle} L^2(\mathbb{R}^3), \mathrm{e}^{\delta\langle x_3\rangle} L^2(\mathbb{R}^3)\right)$. The multiplicity of a resonance $z_0 := z(k_0)$ is defined by*

$$\mathrm{mult}(z_0) := \mathrm{Ind}_{\mathcal{C}}\Big(I + \mathcal{T}_V\big(z(\cdot)\big)\Big), \tag{3.21}$$

$\mathcal{C}$ being a small contour positively oriented containing k_0 as the unique point $k \in D(0,\epsilon)^*$ satisfying $z(k)$ is a resonance of H_V, and $\mathcal{T}_V\big(z(\cdot)\big)$ being defined by Lemma 3.1.

4. Results on the resonances

We establish preliminary results on the resonances we need for the proofs of our main results.

4.1. **A characterisation of the resonances.**

Proposition 4.1. *Let $\mathcal{T}_V(\cdot)$ be defined by Lemma 3.1. Then the following assertions are equivalent:*

(i) $z_0 := z(k_0)$ *is a resonance of* H_V *near zero,*

(ii) -1 *is an eigenvalue of* $\mathcal{T}_V\big(z(k_0)\big)$,

(iii) $\det_2\big(I + \mathcal{T}_V\big(z(k_0)\big)\big) = 0$.

Moreover the multiplicity of z_0 *as zero of* $\det_2\big(I + \mathcal{T}_V(\cdot)\big)$ *coincides with its multiplicity* (3.21) *as resonance of* H_V.

Proof. The equivalence **(i)** $\Leftrightarrow$ **(ii)** follows immediately from

$$\big(I + J|V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}\big)\big(I - J|V|^{1/2}(H_V - z)^{-1}|V|^{1/2}\big) = I. \tag{4.1}$$

The equivalence **(ii)** $\Leftrightarrow$ **(iii)** is a direct consequence of the definition of $\det_2\big(I+\mathcal{T}_V\big(z(k_0)\big)\big)$ given by (1.19) with $q = 2$.

Otherwise since by Lemma 3.1 $\mathcal{T}_V(\cdot)$ is holomorphic on $D(0,\epsilon)^*$ then so is $\det_2\big(I+\mathcal{T}_V(\cdot)\big)$ on $D(0,\epsilon)^*$. Let $\mathrm{m}(z_0)$ be the multiplicity of z_0 as zero of $\det_2\big(I + \mathcal{T}_V(\cdot)\big)$. If $\mathcal{C}'$ is a small contour positively oriented containing z_0 as the unique resonance of H_V near zero then

$$\mathrm{m}(z_0) = ind_{\mathcal{C}'}\Big(\det_2\big(I + \mathcal{T}_V(\cdot)\big)\Big), \tag{4.2}$$

where the RHS of (4.2) is the index defined by (4.42) of the holomorphic function $\det_2\big(I+\mathcal{T}_V(\cdot)\big)$ with respect to the contour $\mathcal{C}'$. Now the equality on the multiplicities claimed in the proposition is an immediate consequence of the equality

$$ind_{\mathcal{C}'}\Big(\det_2\big(I + \mathcal{T}_V(\cdot)\big)\Big) = Ind_{\mathcal{C}}\Big(I + \mathcal{T}_V\big(z(\cdot)\big)\Big),$$

see for instance [6, (2.6)]. This concludes the proof. $\square$

4.2. **Decomposition of the weighted resolvent.**

We split the weighted resolvent $\mathcal{T}_V\big(z(k)\big) := J|V|^{\frac12}\big(H_0 - z(k)\big)^{-1}|V|^{\frac12}$ into a singular part near $k = 0$ and a holomorphic part on the open disk $D(0,\epsilon) := D(0,\epsilon)^* \cup \{0\}$ with values in $\mathcal{S}_2\big(L^2(\mathbb{R}^3)\big)$.

According to (3.3) for $k \in D(0,\epsilon)^*$

$$\mathcal{T}_V\big(z(k)\big) = J|V|^{\frac12}p \otimes \mathscr{R}\big(z(k)\big)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}|V|^{\frac12} + J|V|^{\frac12}\big(H_0 - z(k)\big)^{-1}\mathrm{Q}|V|^{\frac12}. \tag{4.3}$$

Recall that $e_\pm$ are the multiplications operators by the functions $e^{\pm\frac{\gamma}{2}\langle\cdot\rangle}$ respectively. We have

$$J|V|^{\frac12}p \otimes \mathscr{R}\big(z(k)\big)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}|V|^{\frac12} = J|V|^{\frac12}e_+p \otimes e_-\mathscr{R}\big(z(k)\big)e_-\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}e_+|V|^{\frac12}. \tag{4.4}$$

Thanks to (3.4) the integral kernel of $e_-\mathscr{R}\big(z(k)\big)e_-$ is given by

$$e^{-\frac{\gamma}{2}\langle t\rangle}\frac{i\mathrm{e}^{ik|t-t'|}}{2k}e^{-\frac{\gamma}{2}\langle t'\rangle} \tag{4.5}$$

for $k \in D(0,\epsilon)^*$. Then $e_-\mathscr{R}\big(z(k)\big)e_-$ can be decompose as

$$e_-\mathscr{R}\big(z(k)\big)e_- = \frac{1}{k}a + b(k), \tag{4.6}$$

where $a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is the rank-one operator defined by

$$a(u) := \frac{i}{2}\big\langle u, e^{-\frac{\gamma}{2}\langle\cdot\rangle}\big\rangle e^{-\frac{\gamma}{2}\langle\cdot\rangle} \tag{4.7}$$

and $b(k)$ is the operator with integral kernel given by

$$e^{-\frac{\gamma}{2}\langle t\rangle}i\frac{\mathrm{e}^{ik|t-t'|}-1}{2k}e^{-\frac{\gamma}{2}\langle t\rangle}. \tag{4.8}$$

It is easy to remark that $-2ia = c^*c$ where c is the operator defined by (2.5). This together with (4.6) yield for $k \in D(0,\epsilon)^*$ to

$$p \otimes e_-\mathscr{R}\big(z(k)\big)e_- = \pm\frac{i}{2k}p \otimes c^*c + p \otimes s(k), \tag{4.9}$$

where $s(k)$ is the operator acting from $e^{-\frac{\gamma}{2}\langle t\rangle}L^2(\mathbb{R})$ to $e^{\frac{\gamma}{2}\langle t\rangle}L^2(\mathbb{R})$ having the integral kernel

$$\frac{1-\mathrm{e}^{ik|t-t'|}}{2ik}. \tag{4.10}$$

By combining (4.4) with (4.9) we get for $k \in D(0,\epsilon)^*$

$$\begin{aligned} &J|V|^{\frac{1}{2}}p \otimes \mathscr{R}\big(z(k)\big)\begin{pmatrix}1&0\\0&0\end{pmatrix}|V|^{\frac{1}{2}} \\ &= \frac{iJ}{2k}|V|^{\frac{1}{2}}e_+(p\otimes c^*c)\begin{pmatrix}1&0\\0&0\end{pmatrix}e_+|V|^{\frac{1}{2}} + J|V|^{\frac{1}{2}}e_+p\otimes s(k)\begin{pmatrix}1&0\\0&0\end{pmatrix}e_+|V|^{\frac{1}{2}}. \end{aligned} \tag{4.11}$$

That is

$$J|V|^{\frac{1}{2}}p \otimes \mathscr{R}\big(z(k)\big)\begin{pmatrix}1&0\\0&0\end{pmatrix}|V|^{\frac{1}{2}} = \frac{iJ}{k}K^*K + J|V|^{\frac{1}{2}}e_+p\otimes s(k)\begin{pmatrix}1&0\\0&0\end{pmatrix}e_+|V|^{\frac{1}{2}}, \tag{4.12}$$

where K is the operator defined by (2.6). We have then proved the following

Proposition 4.2. *Let V satisfy assumptions* (1.15)-(1.16). *For $k \in D(0,\epsilon)^*$*

$$\mathcal{T}_V\big(z(k)\big) = \frac{iJ}{k}\mathscr{B} + \mathscr{A}(k), \quad \mathscr{B} := K^*K, \tag{4.13}$$

the operator $\mathscr{A}(k) \in \mathcal{S}_2\big(L^2(\mathbb{R}^3)\big)$ being given by

$$\mathscr{A}(k) := J|V|^{\frac{1}{2}}e_+p\otimes s(k)\begin{pmatrix}1&0\\0&0\end{pmatrix}e_+|V|^{\frac{1}{2}} + J|V|^{\frac{1}{2}}\big(H_0 - z(k)\big)^{-1}\mathrm{Q}|V|^{\frac{1}{2}} \tag{4.14}$$

and holomorphic on the open disk $D(0,\epsilon)$ with $s(k)$ defined by (4.9).

Remark 4.1. –

For any $r > 0$ we have

$$\operatorname{Tr} \mathbf{1}_{(r,\infty)} (K^* K) = \operatorname{Tr} \mathbf{1}_{(r,\infty)} (K K^*) = \operatorname{Tr} \mathbf{1}_{(r,\infty)} \big(p \mathbf{W} p \big) \tag{4.15}$$

following (2.9).

Note that the asymptotic expansion of the quantity $\operatorname{Tr} \mathbf{1}_{(r,\infty)} \big(pUp \big)$ is well known once the function $0 \leq U \in L^\infty(\mathbb{R}^2)$ decays like a power, exponentially or is compactly supported:

(A1) If $U \in C^1\big(\mathbb{R}^2\big)$ satisfies $U(x_\perp) = u_0\big(x_\perp/|x_\perp|\big)|x_\perp|^{-m}\big(1 + o(1)\big)$, $|x_\perp| \to \infty$, $0 \not\equiv u_0 \in C^0\big(\mathbb{S}^1, \mathbb{R}_+\big)$, $|\nabla U(x_\perp)| \leq C_1 \langle x_\perp \rangle^{-m-1}$ with m, $C_1 > 0$ constant and if there exists an integrated density of states for the operator $H_1(b)$ then

$$\operatorname{Tr} \mathbf{1}_{(r,\infty)} \big(pUp \big) = C_m r^{-2/m} \big(1 + o(1) \big), \ \ r \searrow 0, \tag{4.16}$$

where $C_m := \frac{b_0}{4\pi} \int_{\mathbb{S}^1} u_0(t)^{2/m} dt$, $\big($see [18, Lemma 3.3]$\big)$.

(A2) If U satisfies $\ln U(x_\perp) = -\mu |x_\perp|^{2\beta}\big(1 + o(1)\big)$, $|x_\perp| \to \infty$ with β, $\mu > 0$ constant then

$$\operatorname{Tr} \mathbf{1}_{(r,\infty)} \big(pUp \big) = \varphi_\beta(r) \big(1 + o(1) \big), \ \ r \searrow 0, \tag{4.17}$$

where for $0 < r < \mathrm{e}^{-1}$

$$\varphi_\beta(r) := \begin{cases} \frac{1}{2} b_0 \mu^{-1/\beta} |\ln r|^{1/\beta} & \text{if } 0 < \beta < 1, \\ \frac{1}{\ln(1 + 2\mu/b_0)} |\ln r| & \text{if } \beta = 1, \\ \frac{\beta}{\beta - 1} \big(\ln |\ln r| \big)^{-1} |\ln r| & \text{if } \beta > 1, \end{cases}$$

$\big($see [18, Lemma 3.4]$\big)$.

(A3) If U is compactly supported and if there exists $C > 0$ constant such that on an open non-empty subset of $\mathbb{R}^2$ $U \geq C$ then

$$\operatorname{Tr} \mathbf{1}_{(r,\infty)} \big(pUp \big) = \varphi_\infty(r) \big(1 + o(1) \big), \ \ r \searrow 0, \tag{4.18}$$

where $\varphi_\infty(r) := \big(\ln |\ln r| \big)^{-1} |\ln r|, \ \ 0 < r < \mathrm{e}^{-1}$, $\big($see [18, Lemma 3.5]$\big)$.

By an evident adaptation of [5, Proof of Corollary 1] we obtain the following corollary summarizing useful properties of the operator $\mathscr{B}$ defined by (4.13). Therefore we omit the proof.

Corollary 4.1. *Let V satisfy assumptions* (1.15)-(1.16). *Then $\mathscr{B} \in \mathcal{S}_1\big(L^2(\mathbb{R}^3)\big)$ and satisfies for $r > 0$ small enough*

$$\operatorname{Tr} \mathbf{1}_{(r,\infty)} (\mathscr{B}) = \mathcal{O}\big(r^{-2/m_\perp} \big). \tag{4.19}$$

For $j \in \mathbb{N}^*$ the operator-valued functions

$$\mathbb{C} \setminus \big(\mp i[0, +\infty[\big) \ni k \mapsto \mathfrak{B}(k) = \mathfrak{B}_j^\pm(k) := \frac{i\mathscr{B}}{k} \left(I \pm \frac{i\mathscr{B}}{k} \right)^{-j} \in \mathcal{S}_1\big(L^2(\mathbb{R}^3)\big) \tag{4.20}$$

are holomorphic and

$$\|\mathfrak{B}(k)\|_{\mathcal{S}_p}^p \leq f(\theta)^{pj}\sigma_p\big(|k|\big), \qquad p = 1,\ 2, \tag{4.21}$$

where $\theta = \operatorname{Arg} k$, $f(\theta) = \big(1 - (\sin\theta)_-\big)^{-\frac{1}{2}}$ *with* $s_- := \max(-s, 0)$ *for* $s \in \mathbb{R}$ *and*

$$\sigma_p(r) := \left\| \frac{\mathscr{B}}{r}\left(I + \frac{\mathscr{B}^2}{r^2}\right)^{-1/2} \right\|_{\mathcal{S}_p}^p = \mathcal{O}\big(r^{-2/m_\perp}\big), \quad r > 0. \tag{4.22}$$

Further for any $r > 0$ *and* $p \geq 1$

$$2^{-p/2}\tilde{n}_p(r) \leq \sigma_p(r) \leq \tilde{n}_p(r) + \operatorname{Tr}\mathbf{1}_{(r,\infty)}(\mathscr{B}) \tag{4.23}$$

with

$$\tilde{n}_p(r) := \left\| \frac{\mathscr{B}}{r}\mathbf{1}_{[0,r]}(\mathscr{B}) \right\|_{\mathcal{S}_p}^p. \tag{4.24}$$

Moreover if the function $\mathbf{W}$ *defined by* (2.3) *satisfies assumption* ***(A1)*** *with* $m > 2$ *then for* $p = 1, 2$ *there exists constants* $C_{m,p}$ *and* $\tilde{C}_{m,p}$ *such that*

$$\begin{cases} \sigma_p(r) = C_{m,p} r^{-2/m}\big(1 + o(1)\big), \\ \tilde{n}_p(r) = \tilde{C}_{m,p} r^{-2/m}\big(1 + o(1)\big), \end{cases} \qquad r \searrow 0. \tag{4.25}$$

Finally if $\mathbf{W}$ *satisfies Assumptions* ***(A2)*** *then*

$$\sigma_p(r) = \varphi_\beta(r)\big(1 + o(1)\big), \quad \tilde{n}_p(r) = o\big(\varphi_\beta(r)\big), \quad r \searrow 0, \tag{4.26}$$

where the functions $\varphi_\beta(r)$, $\beta \in (0, \infty]$ *are defined by* (4.17) *or* (4.18).

4.3. **Upper bounds on the number of resonances.** The next result concerns an upper bound on the number of resonances near zero outside a vicinity of $\big\{z(k) : k \in -iJ[0, +\infty)\big\}$ for potentials V of definite sign $J = \pm$.

Theorem 4.1. *Assume that* V *satisfying assumptions* (1.15)-(1.16) *is of definite sign* J. *Let* $\mathcal{C}_\delta(J)$ *be the sector defined by*

$$\mathcal{C}_\delta(J) := \big\{k \in \mathbb{C} : -\delta J \operatorname{Im}(k) \leq |\operatorname{Re}(k)|\big\}. \tag{4.27}$$

Then for any $\delta > 0$ *there exists* $r_0 > 0$ *such that for any* $0 < r < r_0$

$$\sum_{\substack{z(k) \in \operatorname{Res}(H_V) \\ k \in \{r < |k| < 2r\} \cap \mathcal{C}_\delta(J)}} \operatorname{mult}\big(z(k)\big) = \mathcal{O}\big(|\ln r|\big). \tag{4.28}$$

Proof. Thanks to Proposition 4.2 for $k \in D(0, \epsilon)^*$

$$\mathcal{T}_V\big(z(k)\big) = \frac{iJ}{k}\mathscr{B} + \mathscr{A}(k), \tag{4.29}$$

where $\mathscr{B}$ is a self-adjoint positive operator which does not depend on k while $\mathscr{A}(k) \in \mathcal{S}_2\big(L^2(\mathbb{R}^3)\big)$ is holomorphic near $k = 0$. Since $I + \frac{iJ}{k}\mathscr{B} = \frac{iJ}{k}(\mathscr{B} - iJk)$ then $I + \frac{iJ}{k}\mathscr{B}$ is invertible for $iJk \notin \sigma(\mathscr{B})$ and satisfies

$$\left\| \left(I + \frac{iJ}{k}\mathscr{B} \right)^{-1} \right\| \leq \frac{|k|}{\sqrt{\big(J\mathrm{Im}(k)\big)_+^2 + |\mathrm{Re}(k)|^2}}, \qquad r_+ := \max(r, 0). \tag{4.30}$$

Further it is easy to check that for $k \in \mathcal{C}_\delta(J)$ we have uniformly with respect to $|k| < r_0 \leq \epsilon$

$$\left\| \left(I + \frac{iJ}{k}\mathscr{B} \right)^{-1} \right\| \leq \sqrt{1 + \delta^{-2}}. \tag{4.31}$$

Then using (4.29) we can write

$$I + \mathcal{T}_V\big(z(k)\big) = \big(I + A(k)\big) \left(I + \frac{iJ}{k}\mathscr{B} \right), \tag{4.32}$$

where $A(k)$ is given by

$$A(k) := \mathscr{A}(k) \left(I + \frac{iJ}{k}\mathscr{B} \right)^{-1} \in \mathcal{S}_2\big(L^2(\mathbb{R}^3)\big). \tag{4.33}$$

Otherwise a simple computation allows to obtain

$$\mathcal{T}_V\big(z(k)\big) - A(k) = \big(I + A(k)\big)\frac{iJ}{k}\mathscr{B} \in \mathcal{S}_1\big(L^2(\mathbb{R}^3)\big)$$

since $\mathscr{B} \in \mathcal{S}_1\big(L^2(\mathbb{R}^3)\big)$ by Corollary 4.1. Then if we approximate $A(k)$ by a finite rank-operator in (4.32) and use the formula $\det_2(I + T) = \det(I + T)e^{-\mathrm{Tr}(T)}$ for $T \in \mathcal{S}_1$ we obtain

$$\det{}_2\Big(I + \mathcal{T}_V\big(z(k)\big)\Big) = \det \left(I + \frac{iJ}{k}\mathscr{B} \right) \times \det{}_2\big(I + A(k)\big) e^{-\mathrm{Tr}\left(\mathcal{T}_V\left(z(k)\right) - A(k)\right)}. \tag{4.34}$$

Then for $|k| < r_0$ such that $k \in \mathcal{C}_\delta(J)$ the zeros of $\det_2\big(I + \mathcal{T}_V\big(z(k)\big)\big)$ are those of $\det_2\big(I + A(k)\big)$ with the same multiplicities thanks to Proposition 4.1 and Property (8.3) applied to (4.32).

Estimate (4.31) and the fact that $\mathscr{A}(k)$ is holomorphic near $k = 0$ with values in $\mathcal{S}_2\big(L^2(\mathbb{R}^3)\big)$ imply that the Hilbert-Schmidt norm of $A(k)$ is uniformly bounded with respect to $|k| < r_0$ small enough and $k \in \mathcal{C}_\delta(J)$. So we obtain uniformly with respect to k

$$\det{}_2\big(I + A(k)\big) = \mathcal{O}\left(e^{\mathcal{O}\left(\|A(k)\|_{\mathcal{S}_2}^2\right)} \right) = \mathcal{O}(1). \tag{4.35}$$

In what follows below we prove a corresponding lower bound of (4.35). Identity (4.32) implies that

$$\big(I + A(k)\big)^{-1} = \left(I + \frac{iJ}{k}\mathscr{B} \right) \Big(I + \mathcal{T}_V\big(z(k)\big)\Big)^{-1}. \tag{4.36}$$

With the help of (4.1) we get for $\mathrm{Im}(k^2) > \varsigma > 0$

$$\begin{aligned}\Big\|\Big(I+\mathcal{T}_V\big(z(k)\big)\Big)^{-1}\Big\| &= \mathcal{O}\Big(1+\Big\||V|^{1/2}\big(H_V-z(k)\big)^{-1}|V|^{1/2}\Big\|\Big)\\ &= \mathcal{O}\Big(1+\big|\mathrm{Im}(k^2)\big|^{-1}\Big) = \mathcal{O}\left(\varsigma^{-1}\right).\end{aligned} \tag{4.37}$$

This together with (4.36) yield to

$$\Big\|\big(I+A(k)\big)^{-1}\Big\| = \mathcal{O}\big(s^{-1}\big)\mathcal{O}\big(\varsigma^{-1}\big), \tag{4.38}$$

uniformly with respect to (k,s) such that $0 < s < |k| < r_0$ and $\mathrm{Im}(k^2) > \varsigma > 0$. Let $(\mu_j)_j$ be the sequence of eigenvalues of $A(k)$. We have

$$\begin{aligned}\left|\Big(\det{}_2(I+A(k))\Big)^{-1}\right| &= \big|\det\big((I+A(k))^{-1}e^{A(k)}\big)\big|\\ &\leq \prod_{|\mu_j|\leq\frac{1}{2}}\left|\frac{e^{\mu_j}}{1+\mu_j}\right| \times \prod_{|\mu_j|>\frac{1}{2}}\frac{e^{|\mu_j|}}{|1+\mu_j|}.\end{aligned} \tag{4.39}$$

Using the uniform bound $\|A(k)\|_{\mathcal{S}_2} = \mathcal{O}(1)$ with respect to $|k| < r_0$ small enough and $k \in \mathcal{C}_\delta(J)$ we can prove that the first product is uniformly bounded. On the other hand thanks to (4.38) we have uniformly with respect to (k,s), $0 < s < |k| < r_0$ and $\mathrm{Im}(k^2) > \varsigma > 0$

$$|1+\mu_j|^{-1} = \mathcal{O}\big(s^{-1}\big)\mathcal{O}\big(\varsigma^{-1}\big). \tag{4.40}$$

Therefore using the fact that the second product has a finite number of terms μ_j we deduce from (4.40) that

$$\Big|\det{}_2\big(I+A(k)\big)\Big| \geq Ce^{-C\big(|\ln\varsigma|+|\ln s|\big)}, \tag{4.41}$$

for some $C > 0$ constant. To conclude the proof we need the following Jensen type lemma $\big($see for instance [5, Lemma 6]$\big)$:

Lemma 4.1. *Let Δ be a simply connected sub-domain of $\mathbb{C}$ and let g be a holomorphic function in Δ with continuous extension to $\overline{\Delta}$. Assume there exists $\lambda_0 \in \Delta$ such that $g(\lambda_0) \neq 0$ and $g(\lambda) \neq 0$ for $\lambda \in \partial\Delta$ the boundary of Δ. Let $\lambda_1, \lambda_2, \ldots, \lambda_N \in \Delta$ be the zeros of g repeated according to their multiplicity. Then for any domain $\Delta' \Subset \Delta$ there exists $C' > 0$ such that $N(\Delta', g)$ the number of zeros λ_j of g contained in Δ' satisfies*

$$N(\Delta', g) \leq C'\left(\int_{\partial\Delta}\ln|g(\lambda)|d\lambda - \ln|g(\lambda_0)|\right). \tag{4.42}$$

Consider the domain $\Delta := \big\{k \in D(0,\epsilon)^* : r < |k| < 2r\big\} \cap \mathcal{C}_\delta(J)$ with some $\mathrm{Im}(k_0^2) > \varsigma > 0$, $k_0 \in \Delta$. Then Theorem 4.1 follows immediately by applying the Jensen Lemma 4.1 to the function $D(\cdot) := \det_2\big(I + A(\cdot)\big)$ on Δ together with Proposition 4.1, estimates (4.35)-(4.41). The proof is complete. ☐

For general perturbations V without sign restriction we have the following result:

Theorem 4.2. [19, Theorem 2.1]

Let V satisfy assumptions (1.15)-(1.16)*. Then there exists $0 < r_0 < \epsilon$ small enough such that for any $0 < r < r_0$*

$$\sum_{\substack{z(k)\in \mathrm{Res}(H_V)\\ k\in\{r<|k|<2r\}}} \mathrm{mult}\big(z(k)\big) = \mathcal{O}\Big(\mathrm{Tr}\,\mathbf{1}_{(r,\infty)}\big(p\mathbf{W}p\big)|\ln r|\Big). \tag{4.43}$$

5. Proof of Theorem 2.1: Breit-Wigner approximation

We recall that $N_{\gamma,\zeta}$ is the constant defined by (2.10).

5.1. **Preliminary results.**

Lemma 5.1. *Let V satisfy assumptions* (1.15)-(1.16) *and $\mathcal{T}_V(\cdot)$ be the operator defined by Lemma* (3.1)*. Then on $]-N_{\gamma,\zeta}^2, N_{\gamma,\zeta}^2[\backslash\{0\}$*

$$\xi' = \xi_2' + \frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\left(\partial_z\mathcal{T}_V(\cdot)\right). \tag{5.1}$$

Proof. To get (5.1) thanks to (1.27) and (1.29) it suffices to prove that for any function $f \in C_0^\infty\left(]-N_{\gamma,\zeta}^2, N_{\gamma,\zeta}^2[\backslash\{0\}\right)$

$$\mathrm{Tr}\left(\frac{d}{d\varepsilon}f(H_0+\varepsilon V)_{|\varepsilon=0}\right) = -\frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\mathrm{Im}\mathrm{Tr}\left(\partial_z\mathcal{T}_V(\lambda)\right)d\lambda. \tag{5.2}$$

Recall that by the Helffer-Sjöstrand formula (see for instance [8]) for an analytic extension $\tilde{f} \in C_0^\infty(\mathbb{R}^2)$ of f $\big($*i.e.* $\tilde{f}_{|\mathbb{R}} = f$ and $\bar{\partial}_z\tilde{f}(z) = \mathcal{O}\big(|\mathrm{Im}(z)|^\infty\big)\big)$ we have

$$f(H_0+\varepsilon V) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z\tilde{f}(z)(z-H_0-\varepsilon V)^{-1}L(dz), \tag{5.3}$$

$L(dz)$ being the Lebesgue measure on $\mathbb{C}$. Quantity (5.3) is differentiable with respect to ε and it is easy to check that

$$\frac{d}{d\varepsilon}f(H_0+\varepsilon V)_{|\varepsilon=0} = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z\tilde{f}(z)(z-H_0)^{-1}V(z-H_0)^{-1}L(dz). \tag{5.4}$$

Exploiting the diamagnetic inequality and the boundedness of the magnetic field b it can be checked that for $\pm\mathrm{Im}(z) > 0$ the operator $(z-H_0)^{-1}V(z-H_0)^{-1}$ is of trace class. For $\mathrm{Im}(z) > 0$ by the cyclicity of the trace we have

$$\mathrm{Tr}\Big((z-H_0)^{-1}V(z-H_0)^{-1}\Big) = \mathrm{Tr}\Big(J|V|^{\frac{1}{2}}(z-H_0)^{-2}|V|^{\frac{1}{2}}\Big) = \mathrm{Tr}\Big(\partial_z\mathcal{T}_V(z)\Big) \tag{5.5}$$

and for $\mathrm{Im}(z) < 0$

$$\mathrm{Tr}\Big((z-H_0)^{-1}V(z-H_0)^{-1}\Big) = -\overline{\mathrm{Tr}\Big(\partial_z\mathcal{T}_V(\bar{z})\Big)}. \tag{5.6}$$

Therefore the operator $\frac{d}{d\varepsilon}f(H_0+\varepsilon V)_{|\varepsilon=0}$ is of trace class and using (5.4) we get

$$\begin{aligned}\operatorname{Tr}\left(\frac{d}{d\varepsilon}f(H_0+\varepsilon V)_{|\varepsilon=0}\right) &= -\frac{1}{\pi}\int_{\operatorname{Im}(z)>0}\bar{\partial}_z\tilde{f}(z)\operatorname{Tr}\big(\partial_z\mathcal{T}_V(z)\big)L(dz)\\ &\quad+\frac{1}{\pi}\int_{\operatorname{Im}(z)<0}\bar{\partial}_z\tilde{f}(z)\overline{\operatorname{Tr}\big(\partial_z\mathcal{T}_V(\bar{z})\big)}L(dz).\end{aligned}\tag{5.7}$$

Now (5.2) follows immediately from (5.7) using the Green formula. □

For further use we recall complex analysis results due to J. Sjöstrand summarized in the following

Proposition 5.1. [21], [22]

Let $\Omega\subset\mathbb{C}$ be a simply connected domain satisfying $\Omega\cap\mathbb{C}^+\neq\emptyset$. Let $z\mapsto F(z,h)$, $0<h<h_0$ be a family of holomorphic functions in Ω having at most a finite number $N(h)\in\mathbb{N}^*$ of zeros in Ω. Suppose that

$$F(z,h)=\mathcal{O}(1)e^{\mathcal{O}(1)N(h)},\quad z\in\Omega,\tag{5.8}$$

and that there exists constants C, $\varsigma>0$ with $\Omega_\varsigma:=\big\{z\in\mathbb{C}:\operatorname{Im}(z)>\varsigma\big\}\neq\emptyset$ such that

$$|F(z,h)|\geq e^{-CN(h)},\quad z\in\Omega_\varsigma.\tag{5.9}$$

Then for any $\tilde{\Omega}\Subset\Omega$ there exists $g(\cdot,h)$ holomorphic in Ω such that

$$F(z,h)=\prod_{j=1}^{N(h)}(z-z_j)e^{g(z,h)},\quad \frac{d}{dz}g(z,h)=\mathcal{O}\big(N(h)\big),\quad z\in\tilde{\Omega},\tag{5.10}$$

where the z_j are the zeros of $F(z,h)$ in Ω.

In the next proposition the domains $\mathscr{W}_\pm\Subset\Omega_\pm$ and the intervals $I_\pm$ are introduced in Section 2 just after (2.10).

Proposition 5.2. *Assume that V satisfies assumptions* (1.15)-(1.16)*. Let $\mathscr{W}_\pm\Subset\Omega_\pm$ and $I_\pm$ be as above. Then there exists $r_0>0$ and holomorphic functions $g_\pm$ in $\Omega_\pm$ satisfying for any $\mu\in rI_\pm$*

$$\begin{aligned}\xi_2'(\mu) &= \frac{1}{\pi r}\operatorname{Im}g_\pm'\left(\frac{\mu}{r},r\right)+\sum_{\substack{w\in\operatorname{Res}(H_V)\cap r\Omega_\pm\\ \operatorname{Im}(w)\neq 0}}\frac{\operatorname{Im}(w)}{\pi|\mu-w|^2}\\ &\quad-\sum_{w\in\operatorname{Res}(H_V)\cap rI_\pm}\delta(\mu-w)-\frac{1}{\pi}\operatorname{Im}\operatorname{Tr}\big(\partial_z\mathcal{T}_V(\mu)\big),\end{aligned}\tag{5.11}$$

where the functions $g_\pm(\cdot,r)$ satisfy

$$\begin{aligned}g_\pm(z,r) &= \mathcal{O}\left[\operatorname{Tr}\mathbf{1}_{(s_1\sqrt{r},\infty)}\big(p\mathbf{W}p\big)|\ln r|+\tilde{n}_1\left(\frac{1}{2}s_1\sqrt{r}\right)+\tilde{n}_2\left(\frac{1}{2}s_1\sqrt{r}\right)\right]\\ &= \mathcal{O}\left(|\ln r|r^{-1/m_\perp}\right),\end{aligned}\tag{5.12}$$

uniformly with respect to $0 < r < r_0$ *and* $z \in \Omega_\pm$ *with* $\tilde{n}_q$, $q = 1, 2$ *defined by* (4.24).

Proof. The first step consists to reduce the study of the zeros of the 2-regularized perturbation determinant to that of a suitable holomorphic function in $\Omega_\pm$ satisfying the assumptions of Proposition 5.1.

By Proposition 4.2 for $0 < s < |k| \leq s_0 < \epsilon$

$$\mathcal{T}_V\big(z(k)\big) = \frac{iJ}{k}\mathscr{B} + \mathscr{A}(k).$$

The operator-valued function $k \mapsto \mathscr{A}(k)$ is analytic near zero with values in $\mathcal{S}_2\big(L^2(\mathbb{R}^3)\big)$. Then for s_0 small enough there exists $\mathscr{A}_0$ a finite-rank operator independent of k and $\tilde{\mathscr{A}}(k)$ analytic near zero satisfying $\|\tilde{\mathscr{A}}(k)\| < \frac{1}{4}$, $|k| < s_0$ such that

$$\mathscr{A}(k) = \mathscr{A}_0 + \tilde{\mathscr{A}}(k). \tag{5.13}$$

Consider the decomposition

$$\mathscr{B} = \mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}) + \mathscr{B}\mathbf{1}_{]\frac{1}{2}s,\infty[}(\mathscr{B}). \tag{5.14}$$

Obviously $\Big\|(iJ/k)\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\Big\| < \frac{3}{4}$ for $0 < s < |k| < s_0$. Then

$$I + \mathcal{T}_V\big(z(k)\big) = \big(I + \mathscr{K}(k,s)\big)\left(I + \frac{iJ}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\right), \tag{5.15}$$

where $K(k,s)$ is given by

$$\mathscr{K}(k,s) := \left(\frac{iJ}{k}\mathscr{B}\mathbf{1}_{]\frac{1}{2}s,\infty[}(\mathscr{B}) + \mathscr{A}_0\right)\left(I + \frac{iJ}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\right)^{-1}. \tag{5.16}$$

Its rank is of order

$$O\left(\operatorname{Tr}\mathbf{1}_{(\frac{1}{2}s,\infty)}(\mathscr{B}) + 1\right) = \mathcal{O}\left(\operatorname{Tr}\mathbf{1}_{(s,\infty)}\big(p\mathbf{W}p\big) + 1\right) \tag{5.17}$$

according to (4.15) and moreover its norm is bounded by $\mathcal{O}\left(s^{-1}\right)$ for $0 < s < |k| < s_0$. Since $\|(iJ/k)\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\| < 1$ for $0 < s < |k| < s_0$ then

$$\det\left(\left(I + \frac{iJ}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\right)e^{-T_V\left(z(k)\right)}\right) \neq 0. \tag{5.18}$$

Therefore the zeros of $\det_2\big(I + \mathcal{T}_V\big(z(k)\big)\big)$ are those of

$$\mathscr{D}(k,s) := \det\big(I + \mathscr{K}(k,s)\big) \tag{5.19}$$

with the same multiplicities thanks to Proposition 4.1 and Property (8.3) applied to (5.15). The above properties of $\mathscr{K}(k,s)$ imply that

$$\begin{aligned}\mathscr{D}(k,s) &= \prod_{j=1}^{\mathcal{O}\left(\operatorname{Tr}\mathbf{1}_{(s,\infty)}(p\mathbf{W}p)+1\right)} \big(1+\lambda_j(k,s)\big) \\ &= \mathcal{O}(1)\exp\Big(\mathcal{O}\big(\operatorname{Tr}\mathbf{1}_{(s,\infty)}\big(p\mathbf{W}p\big)+1\big)|\ln s|\Big)\end{aligned} \tag{5.20}$$

for $0<s<|k|<s_0$, where the $\lambda_j(k,s)$ are the eigenvalues of $\mathscr{K} := \mathscr{K}(k,s)$ satisfying $|\lambda_j(k,s)| = \mathcal{O}\left(s^{-1}\right)$. If $\operatorname{Im}(k^2) > \varsigma > 0$ with $0<s<|k|<s_0$ then

$$\mathscr{D}(k,s)^{-1} = \det\big(I+\mathscr{K}\big)^{-1} = \det\big(I-\mathscr{K}(I+\mathscr{K})^{-1}\big).$$

Thus with the help of (4.37) we can show similarly to (5.20) that

$$|\mathscr{D}(k,s)| \geq C\exp\Big(-C\big(\operatorname{Tr}\mathbf{1}_{(s,\infty)}\big(p\mathbf{W}p\big)+1\big)\big(|\ln\varsigma|+|\ln s|\big)\Big). \tag{5.21}$$

Now for $\mathscr{D}(k,s)$ defined by (5.19) fix $0<s_1<\sqrt{\operatorname{dist}\big(\Omega_\pm,0\big)}$ and consider the functions

$$F_\pm : z\in\Omega_\pm \mapsto \mathscr{D}\left(\sqrt{r}\sqrt{z},\sqrt{r}s_1\right) \tag{5.22}$$

where

$$\sqrt{z} = \begin{cases} \sqrt{\rho}e^{i\frac{\theta}{2}} & \text{if} \quad z=\rho e^{i\theta}\in\Omega_+, \\ i\sqrt{\rho}e^{-i\frac{\theta}{2}} & \text{if} \quad z=-\rho e^{-i\theta}\in\Omega_-. \end{cases} \tag{5.23}$$

The functions $F_\pm$ are holomorphic in $\Omega_\pm$ and according to Proposition 4.1 $\tilde\omega$ is a zero of $F_\pm$ if and only if $\omega=\tilde\omega r$ is a resonance of H_V. Then by Proposition 5.1 applied to $F=F_+$ and $F(z)=\overline{F_-(-\bar z)}$ with $h=r$, $N(r)=\operatorname{Tr}\mathbf{1}_{(s_1\sqrt{r},\infty)}\big(p\mathbf{W}p\big)|\ln r|$ there exists holomorphic functions $g_{0,\pm}$ in $\Omega_\pm$ satisfying for any $z\in\Omega_\pm$

$$\mathscr{D}_\pm\left(\sqrt{r}\sqrt{z},\sqrt{r}s_1\right) = \prod_{w\in\operatorname{Res}(H_V)\cap r\Omega_\pm}\left(\frac{zr-\omega}{r}\right)e^{g_{0,\pm}(z,r)} \tag{5.24}$$

with

$$\frac{d}{dz}g_{0,\pm}(z,r) = \mathcal{O}\left(\operatorname{Tr}\mathbf{1}_{(s_1\sqrt{r},\infty)}\big(p\mathbf{W}p\big)|\ln r|\right), \tag{5.25}$$

uniformly with respect to $z\in\mathscr{W}_\pm$.

From above (5.15)-(5.19) we know that for $z=z\big(\sqrt{r}k\big)$, $0<s_1<|k|<s_0$

$$\begin{aligned}&\det{}_2\big(I+\mathcal{T}_V(z)\big) = \\ &\mathscr{D}\left(\sqrt{r}k,\sqrt{r}s_1\right)\det\left(\left(I+\frac{iJ}{\sqrt{r}k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B})+\tilde{\mathscr{A}}(\sqrt{r}k)\right)e^{-T_V(z)}\right).\end{aligned} \tag{5.26}$$

By setting

$$\mathfrak{A}(k) := \frac{iJ}{\sqrt{r}k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B})+\tilde{\mathscr{A}}(\sqrt{r}k)$$

we deduce from (5.15) that $\mathcal{T}_V(z) - \mathfrak{A}(k)$ is a finite-rank operator thanks to the properties of the operator $\mathscr{K}\big(\sqrt{r}k, \sqrt{r}s_1\big)$ given by (5.16). Then as in (4.34) we can prove that

$$\det\Big(\Big(I + \frac{iJ}{\sqrt{r}k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B}) + \tilde{\mathscr{A}}(\sqrt{r}k)\Big)e^{-T_V(z)}\Big) = \det{}_2\big(I + \mathfrak{A}(k)\big)e^{-\mathrm{Tr}\big(T_V(z)-\mathfrak{A}(k)\big)} \tag{5.27}$$

with $\det_2\big(I + \mathfrak{A}(k)\big) \neq 0$ since $\|\mathfrak{A}(k)\| < 1$ for $0 < s_1 < |k| < s_0$. The holomorphicity of $\tilde{\mathscr{A}}(k)$ with values in $\mathcal{S}_2\big(L^2(\mathbb{R}^3)\big)$ combined with (4.24) of Corollary 4.1 imply that

$$\|\mathfrak{A}(k)\|_2^2 = \mathcal{O}\left(\tilde{n}_2\left(\frac{1}{2}\sqrt{r}s_1\right)\right). \tag{5.28}$$

Then we have

$$\det{}_2\big(I + \mathfrak{A}(k)\big) = \mathcal{O}(1)e^{\mathcal{O}(1)\tilde{n}_2\left(\frac{1}{2}\sqrt{r}s_1\right)}. \tag{5.29}$$

On the other hand it can be also checked that

$$\det{}_2\big(I + \mathfrak{A}(k)\big)^{-1} = \det{}_2\left(I - \mathfrak{A}(k)\big(I + \mathfrak{A}(k)\big)^{-1}\right) = \mathcal{O}(1)e^{\mathcal{O}(1)\tilde{n}_2\left(\frac{1}{2}\sqrt{r}s_1\right)}. \tag{5.30}$$

Then Proposition 5.1 implies that there exists $g_1(\cdot, r)$ holomorphic in $\Omega_\pm$ such that

$$\det{}_2\big(I + \mathfrak{A}(k)\big) = e^{g_1(z,r)} \tag{5.31}$$

with

$$\frac{d}{dz}g_1(z,r) = \mathcal{O}\left(\tilde{n}_2\left(\frac{1}{2}\sqrt{r}s_1\right)\right), \tag{5.32}$$

uniformly with respect to $z \in \mathscr{W}_\pm$. Therefore according to definition (1.28) of ξ_2 and by combining (5.26), (5.24), (5.27) with (5.31) we get for $\mu = z\big(\sqrt{r}k\big) = rk^2 \in r(\Omega_\pm \cap \mathbb{R})$

$$\begin{aligned}\xi_2'(\mu) = \frac{1}{\pi r}\mathrm{Im}\,\partial_\lambda(g_{0,\pm} + g_1)\left(\frac{\mu}{r}, r\right) + \sum_{\substack{w\in\mathrm{Res}(H_V)\cap r\Omega_\pm \\ \mathrm{Im}(w)\neq 0}} \frac{\mathrm{Im}(w)}{\pi|\mu - w|^2} - \sum_{w\in\mathrm{Res}(H_V)\cap rI_\pm} \delta(\mu - w) \\ + \frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\left(\frac{1}{2k}\partial_k\left(\frac{iJ}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B}) + \tilde{\mathscr{A}}(k)\right) - \partial_z\mathcal{T}_V(\mu + i0)\right)\end{aligned} \tag{5.33}$$

with

$$k = \begin{cases} \sqrt{\mu} & \text{if} \quad \mu > 0, \\ i\sqrt{-\mu} & \text{if} \quad \mu < 0. \end{cases} \tag{5.34}$$

By (4.24) of Corollary 4.1

$$\mathrm{Tr}\left(\frac{1}{2k}\partial_k\left(\frac{iJ}{k}\mathscr{B}\mathbf{1}_{[0,\frac{1}{2}s_1\sqrt{r}]}(\mathscr{B})\right)\right) = -\frac{iJs_1\sqrt{r}}{4k^3}\tilde{n}_1\left(\frac{1}{2}\sqrt{r}s_1\right). \tag{5.35}$$

Thanks to Lemma 3.1 $\partial_z \mathcal{T}_V(z)$ is of trace class. Then since $\mathscr{B} \in \mathcal{S}_1\big(L^2(\mathbb{R}^3)\big)$ the operator

$$\partial_k \tilde{\mathscr{A}}(k) = \partial_k \mathscr{A}(k) = \partial_k \Big(\mathcal{T}_V\big(z(k)\big) - \frac{iJ}{k}\mathscr{B} \Big) \tag{5.36}$$

is of trace class. Moreover the definition (4.14) of $\mathscr{A}(k)$ implies that
(5.37)

$$\mathrm{Tr}\left(\frac{1}{2k}\partial_k \mathscr{A}(k) \right) = \mathrm{Tr}\left(J|V|^{\frac{1}{2}} \big(H_0 - k^2\big)^{-2} \mathrm{Q}|V|^{\frac{1}{2}} \right) = \mathrm{Tr}\left(J|V|^{\frac{1}{2}} \big(H_0 - \mu\big)^{-2} \mathrm{Q}|V|^{\frac{1}{2}} \right).$$

By setting $g_\pm = g_{0,\pm} + g_1 + g_2$ with

$$g_2(z) = \frac{iJs_1}{2\sqrt{z}} \tilde{n}_1 \left(\frac{1}{2}\sqrt{r} s_1 \right), \tag{5.38}$$

where $\sqrt{z}$ is defined on $\Omega_\pm$ by (5.23) we get the desired conclusion. $\square$

The representation of the SSF near zero can be specified if the potential V is of definite sign $J = sign(V)$. According to Remark 2.1 in the next proposition the case " $-$ " is with respect the definite sign $J = +$.

Proposition 5.3. *Assume the assumptions of Theorem 2.1 with V of definite sign $J = sign(V)$. Then for $\lambda \in rI_\pm$ (5.11) holds with*

$$\frac{1}{r}\mathrm{Im}\, g'_\pm \left(\frac{\lambda}{r}, r \right) = \frac{1}{r}\mathrm{Im}\, \tilde{g}'_\pm \left(\frac{\lambda}{r}, r \right) + \mathrm{Im}\, \tilde{g}'_{1,\pm}(\lambda) + \mathbf{1}_{(0,N^2_{\gamma,\zeta})}(\lambda) J \phi'(\lambda), \tag{5.39}$$

where the function ϕ is defined by

$$\phi(\lambda) := \mathrm{Tr}\left(\arctan \frac{K^*K}{\sqrt{\lambda}} \right) = \mathrm{Tr}\left(\arctan \frac{p\mathbf{W}p}{2\sqrt{\lambda}} \right), \tag{5.40}$$

the functions $z \mapsto \tilde{g}_\pm(z,r)$ being holomorphic in $\Omega_\pm$ and satisfying

$$\tilde{g}_\pm(z,r) = \mathcal{O}\big(|\ln r|\big), \tag{5.41}$$

uniformly with respect to $0 < r < r_0$ and $z \in \Omega_\pm$. The functions $z \mapsto \tilde{g}_{1,\pm}(z)$ are holomorphic in $\pm]0, N^2_{\gamma,\zeta}[e^{\pm i]-2\theta_0,2\varepsilon_0[}$ and there exists a positive constant C_{θ_0} depending on θ_0 such that

$$|\tilde{g}_{1,\pm}(z)| \leq C_{\theta_0} \sigma_2 \left(\sqrt{|z|} \right)^{\frac{1}{2}} \tag{5.42}$$

for $z \in \pm]0, N^2_{\gamma,\zeta}[e^{\pm i]-2\theta_0,2\varepsilon_0[}$, where the quantity $\sigma_2(\cdot)$ is defined by (4.22).

Proof. We use notations of Subsection 4.3. Hence for $z = z\big(\sqrt{r}k\big)$, $0 < s_1 < |k| < s_0$ and $k \in \mathcal{C}_\delta(J)$ (4.34) implies that

$$\det{}_2\big(I + \mathcal{T}_V(z)\big) = \det\left(I + \frac{iJ}{\sqrt{r}k}\mathscr{B} \right) \times \det{}_2\big(I + A(\sqrt{r}k)\big) e^{-\mathrm{Tr}\left(\mathcal{T}_V(z) - A(\sqrt{r}k)\right)}, \tag{5.43}$$

where $A(\sqrt{r}k)$ is given by (4.33) with k replaced by $\sqrt{r}k$. Then as in the previous proof by applying Proposition 5.1 to $\det_2\big(I + A(\sqrt{r}\sqrt{\cdot})\big)$ in $\Omega_\pm$ taking into account (4.35) and (4.41) we get

$$\det_2\big(I + A(\sqrt{r}\sqrt{z})\big) = \prod_{w\in \mathrm{Res}(H_V)\cap r\Omega_\pm} \left(\frac{zr-\omega}{r}\right) e^{\tilde{g}_\pm(z,r)}, \tag{5.44}$$

where $\tilde{g}_\pm$ is holomorphic in $\Omega_\pm$ such that

$$\frac{d}{dz}\tilde{g}_\pm(z,r) = \mathcal{O}\left(|\ln r|\right), \tag{5.45}$$

uniformly with respect to $z \in \mathscr{W}_\pm$. Then according to definition (1.28) of ξ_2 and by combining (5.43)-(5.44) we get for $\mu = z\big(\sqrt{r}k\big) = rk^2 \in r(\Omega_\pm \cap \mathbb{R})$

(5.46)
$$\xi_2'(\mu) = \frac{1}{\pi r}\mathrm{Im}\,\partial_\lambda \tilde{g}_\pm\left(\frac{\mu}{r}, r\right) + \sum_{\substack{w\in \mathrm{Res}(H_V)\cap r\Omega_\pm \\ \mathrm{Im}(w)\neq 0}} \frac{\mathrm{Im}(w)}{\pi|\mu - w|^2} - \sum_{w\in \mathrm{Res}(H_V)\cap rI_\pm} \delta(\mu - w)$$
$$+ \frac{1}{2k\pi}\mathrm{Im}\mathrm{Tr}\left(\left(I + \frac{iJ}{k}\mathscr{B}\right)^{-1} \partial_k \left(\frac{iJ}{k}\mathscr{B}\right)\right) - \frac{1}{\pi}\mathrm{Im}\mathrm{Tr}\left(\partial_z \mathcal{T}_V(\mu + i0) - \frac{1}{2k}\partial_k A(k)\right),$$

where k is defined by (5.34).

By Lemma 3.1 $\partial_z \mathcal{T}_V(z)$ is of trace class. Then as in (5.36) accordingly to definition (4.33) of $A(k)$

$$\partial_k A(k) = \partial_k \mathscr{A}(k) - \partial_k \left(\mathscr{A}(k)\frac{iJ}{k}\mathscr{B}\left(I + \frac{iJ}{k}\mathscr{B}\right)^{-1}\right) \tag{5.47}$$

is of trace class. For the first term of the RHS of (5.47) equality (5.37) holds. For the second term we have

$$\mathrm{Im}\frac{1}{2k}\mathrm{Tr}\,\partial_k \left(\mathscr{A}(k)\frac{iJ}{k}\mathscr{B}\left(I + \frac{iJ}{k}\mathscr{B}\right)^{-1}\right) = \mathrm{Im}\frac{1}{2k}\partial_k\big(\tilde{g}_{1,\pm}(k^2)\big), \tag{5.48}$$

where $\tilde{g}_{1,\pm}$ is the holomorphic function given by

$$\tilde{g}_{1,\pm}(z) := \mathrm{Tr}\left(\mathscr{A}(\sqrt{z})\frac{iJ}{\sqrt{z}}\mathscr{B}\left(I + \frac{iJ}{\sqrt{z}}\mathscr{B}\right)^{-1}\right) \tag{5.49}$$

satisfying bound (5.42) by Corollary 4.1.

For the fourth term of the RHS of (5.46) we have

$$
\begin{aligned}
&\frac{1}{2k}\mathrm{ImTr}\left(\left(I+\frac{iJ}{k}\mathscr{B}\right)^{-1}\partial_k\left(\frac{iJ}{k}\mathscr{B}\right)\right)\\
&\qquad\qquad= -\frac{1}{2k^2}\mathrm{ImTr}\left(\frac{iJ}{k}\mathscr{B}\left(I+\frac{iJ}{k}\mathscr{B}\right)^{-1}\right)\\
&\qquad\qquad= \begin{cases} 0 & \text{if} \quad Jk\in i\mathbb{R}^+,\\ -\frac{1}{2k^2}\mathrm{Tr}\left(\frac{J}{k}\mathscr{B}\left(I+\frac{\mathscr{B}^2}{k^2}\right)^{-1}\right) = J\Phi'(k^2) & \text{if} \quad k\in\mathbb{R}.\end{cases}
\end{aligned}
\tag{5.50}
$$

Then Proposition 5.3 follows. ☐

5.2. **Back to the proof of Theorem 2.1.** It follows immediately by combining Lemma 5.1 with Propositions 5.2-5.3.

6. Proof of Theorem 2.2: Singularity at the low ground state

We begin by applying Theorem 2.1 on intervals of the form $r_n[1,2]$, $r_n = 2^n\lambda$ with $\lambda>0$ small enough. Hence for Ω_+ a complex neighbourhood of $[1,2]$ and $\mu\in r_n[1,2]$ we have

$$
\begin{aligned}
\xi'(\mu) = \frac{1}{r_n\pi}\mathrm{Im}\,\tilde g'_\pm\left(\frac{\mu}{r_n},r_n\right) &+ \sum_{\substack{w\in\mathrm{Res}(H_V)\cap r_n\Omega_+\\ \mathrm{Im}(w)\neq 0}}\frac{\mathrm{Im}(w)}{\pi|\mu-w|^2}\\
&- \sum_{w\in\mathrm{Res}(H_V)\cap r_n[1,2]}\delta(\mu-w) + \frac{1}{\pi}\left(J\phi' + \mathrm{Im}\,\tilde g'_{1,\pm}\right)(\mu).
\end{aligned}
\tag{6.1}
$$

By Theorem 4.1 there exists at most $\mathcal{O}\big(|\ln r_n|\big)$ resonances in $r_n\Omega_+$. Then by integrating (6.1) on $r_n[1,2]$ we obtain

$$
\xi(r_{n+1}) - \xi(r_n) = \frac{1}{\pi}\big[\mathrm{Im}\,\tilde g_\pm(\cdot,r_n)\big]_1^2 + \mathcal{O}\big(|\ln r_n|\big) + \frac{1}{\pi}\big[J\phi + \mathrm{Im}\,\tilde g_{1,\pm}\big]_{r_n}^{r_{n+1}}.
\tag{6.2}
$$

Now choose $N\in\mathbb{N}$ such that $\frac{N^2_{\gamma,\zeta}}{4}\le\lambda 2^{N+1}\le\frac{N^2_{\gamma,\zeta}}{2}$. Then taking the sum in (6.2) and exploiting the fact that in $\frac{N^2_{\gamma,\zeta}}{2}\left[\frac{1}{2},1\right]$ the functions ξ, Φ, $\tilde g_{1,\pm}$ are uniformly bounded together with $\tilde g_\pm(\cdot,r_n) = \mathcal{O}\big(|\ln r_n|\big)$ we get

$$
\xi(\lambda) = \frac{J}{\pi}\Phi(\lambda) + \frac{1}{\pi}\mathrm{Im}\,\tilde g_{1,\pm}(\lambda) + \sum_{n=0}^{N}\mathcal{O}\big(|\ln 2^n\lambda|\big) + \mathcal{O}(1).
\tag{6.3}
$$

Since $N = \mathcal{O}\big(|\ln\lambda|\big)$ and $\tilde g_{1,\pm}$ satisfies (2.19) then (6.3) implies that for λ small enough

$$
\left|\xi(\lambda) - \frac{J}{\pi}\Phi(\lambda)\right| \le C|\ln\lambda|^2 + C\sigma_2\left(\sqrt{\lambda}\right)^{\frac{1}{2}}
\tag{6.4}
$$

for some $C > 0$ constant. For a Hilbert-Schmidt operator L on $\mathscr{H}$ we have $\|L\|_{\mathcal{S}_2}^2 = \operatorname{Tr}(LL^*)$. This together with the elementary inequality

$$\frac{u^2}{1+u^2} \leq \arctan u, \quad u \geq 0$$

imply that $\sigma_2\left(\sqrt{\lambda}\right) \leq \Phi(\lambda)$, which completes the proof.

7. Proof of Theorem 2.3: Local trace formula

For simplicity of notation we ignore in the proof the dependence on the subscript $\pm$. Let $\tilde{\psi} \in C_0^\infty\big(\Omega\big)$ be an almost analytic extension of ψ such that $\tilde{\psi} = 1$ on $\mathcal{W}$ and

$$\operatorname{supp} \bar{\partial}_z \tilde{\psi} \subset \Omega \setminus \mathcal{W}. \tag{7.1}$$

By Applying (1.27) and Theorem 2.1 we get

$$\begin{aligned} \operatorname{Tr}\left[(\psi f)\left(\frac{H_V}{r}\right) - (\psi f)\left(\frac{H_0}{r}\right)\right] &= -\left\langle \xi'(\lambda), (\psi f)\left(\frac{\lambda}{r}\right)\right\rangle \\ &= \sum_{w \in \operatorname{Res}(H_V) \cap r \operatorname{supp} \psi} (\psi f)\left(\frac{w}{r}\right) - \frac{1}{\pi}\int (\psi f)\left(\frac{\lambda}{r}\right) \operatorname{Im} g'\left(\frac{\lambda}{r}, r\right)\frac{d\lambda}{r} \\ &+ \sum_{\substack{w \in \operatorname{Res}(H_V) \cap r \operatorname{supp} \psi \\ \operatorname{Im}(w) \neq 0}} \frac{1}{2\pi i}\int (\psi f)\left(\frac{\lambda}{r}\right)\left(\frac{1}{\lambda - \overline{w}} - \frac{1}{\lambda - w}\right) d\lambda. \end{aligned} \tag{7.2}$$

Using the Green formula and (2.15) on $\operatorname{supp}\tilde{\psi}$ we can estimate the integral involving g'. On the other hand for $w \in \mathbb{C}_- := \big\{z \in \mathbb{C} : \operatorname{Im}(z) < 0\big\}$ by applying the Green formula we get

$$-\frac{1}{\pi}\int_{\mathbb{C}_-} \bar{\partial}_z \tilde{\psi}(z) \frac{1}{z-w} L(dz) + \tilde{\psi}(w) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \tilde{\psi}(\lambda)\frac{1}{\lambda - w} d\lambda \tag{7.3}$$

and

$$-\frac{1}{\pi}\int_{\mathbb{C}_-} \bar{\partial}_z \tilde{\psi}(z) \frac{1}{z-\overline{w}} L(dz) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \tilde{\psi}(\lambda)\frac{1}{\lambda - \overline{w}} d\lambda. \tag{7.4}$$

Since f is holomorphic then with the help of the above formulas and using the fact that $\tilde{\psi} = \psi$ on $\mathbb{R}$ the third term of the RHS of (7.2) is equal to

$$\begin{aligned} &\sum_{w \in \operatorname{Res}(H_V), \operatorname{Im}(w) \neq 0} (\tilde{\psi} f)\left(\frac{w}{r}\right) \\ &\qquad + \sum_{\substack{w \in \operatorname{Res}(H_V) \cap r \operatorname{supp} \tilde{\psi} \\ \operatorname{Im}(w) \neq 0}} \frac{1}{\pi r}\int_{\mathbb{C}_-} (\bar{\partial}_z \tilde{\psi})\left(\frac{z}{r}\right) f\left(\frac{z}{r}\right)\left(\frac{1}{z-\overline{w}} - \frac{1}{z-w}\right) L(dz). \end{aligned} \tag{7.5}$$

Now by using Theorem 4.2 in Ω and the elementary inequality [17, (5.3)]

$$\int_{\Omega_1} \frac{1}{|z-w|} L(dz) \le 2\sqrt{2\pi \mathrm{vol}(\Omega)} \tag{7.6}$$

we get the result.

8. Appendix

We recall in this subsection the notion of index (with respect to a positively oriented contour) of a holomorphic function and a finite meromorphic operator-valued function, see for instance [6, Definition 2.1].

If a function f is holomorphic in a neighbourhood of a contour γ its index with respect to γ is defined by

$$ind_\gamma\, f := \frac{1}{2i\pi} \int_\gamma \frac{f'(z)}{f(z)} dz. \tag{8.1}$$

Let us point out that if f is holomorphic in a domain Ω with $\partial\Omega = \gamma$ then thanks to the residues theorem $\mathrm{ind}_\gamma\, f$ coincides with the number of zeros of f in Ω taking into account their multiplicity.

Let $D \subseteq \mathbb{C}$ be a connected domain, $Z \subset D$ be a pure point and closed subset and $A : \overline{D}\backslash Z \longrightarrow \mathrm{GL}(E)$ a be finite meromorphic operator-valued function which is Fredholm at each point of Z. The index of A with respect to the contour $\partial\Omega$ is defined by

$$Ind_{\partial\Omega}\, A := \frac{1}{2i\pi} \mathrm{Tr} \int_{\partial\Omega} A'(z)A(z)^{-1} dz = \frac{1}{2i\pi} \mathrm{Tr} \int_{\partial\Omega} A(z)^{-1}A'(z) dz. \tag{8.2}$$

The following properties are well known:

$$Ind_{\partial\Omega}\, A_1A_2 = Ind_{\partial\Omega}\, A_1 + Ind_{\partial\Omega}\, A_2; \tag{8.3}$$

for $K(z)$ a trace class operator

$$Ind_{\partial\Omega}\, (I+K) = ind_{\partial\Omega}\, \det{(I+K)}. \tag{8.4}$$

We refer for instance [10, Chap. 4] for more details.

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